ON THE METRIC GEOMETRY OF THE PLANE N-LINE*

BY

F. MORLEY

The relations which n lines of a plane exhibit, when considered in relation to the circular points, have not received systematic attention since the important memoirs by CLIFFORD, *On Miquel's theorem*, Works, p. 51, and by KANTOR, Wiener Berichte, vols. 76, 78.

In what follows I shall apply certain notions which are fundamental in the geometric treatment of the theory of functions, and especially the notion of mapping. In the rational curves which alone are considered here, in place of expressing the Cartesian coördinates X and Y as real rational algebraic functions of a real parameter, we express $X + iY$ or x as a rational algebraic function of a parameter t, which it is convenient to make move on the unit circle. Thus uniformly the various numbers t are of absolute value 1; for shortness they will be called *turns*. Curves then are here considered as maps of the unit circle; and in general one equation, e. g.:

$$x = a_1 - 2a_2 t + a_3 t^2,$$

is sufficient. But of course such an equation carries with it the conjugate equation, which we write:

$$y = b_1 - 2b_2/t + b_3/t^2.$$

The coördinates x, y or $X + iY, X - iY$ are called the circular coördinates of the point x.

§ 1. *The fixing of a line.*

To name a line it is enough to name the reflexion of the origin in that line. Thus by naming a point x_1 we name at the same time the line whose points are equidistant from 0 and from x_1.

To map the line on the unit circle we may write:

$$x = \frac{x_1 t_1}{t_1 - t}, \tag{1}$$

* Presented to the Society February 24, 1900. Received for publication February 5, 1900.

46966

where t_1 is an arbitrarily given turn. For in general an equation linear in both x and t maps the unit circle on another circle; but here x is ∞ when t is t_1, and the circle becomes a line.

Replace for a moment t by any complex number z. When $z = \infty$, $x = 0$; and when $z = 0$, $x = x_1$. Now 0 and ∞ are inverse points of the unit circle; and it is an elementary theorem—but of considerable use—that inverse points map into inverse points. Here then 0 and x_1 are inverse points of the line.

The turn t_1 is at our disposal. We shall fix it by the condition:

$$x_1 t_1 = y_1, \tag{2}$$

where y_1 is the conjugate of x_1.

The conjugate of (1) is

$$y = \frac{y_1 t}{t - t_1};$$

whence the equation of the line in circular coördinates is

$$x/x_1 + y/y_1 = 1,$$

or

$$xt_1 + y = x_1 t_1. \tag{3}$$

So when we are considering n lines we take them as given by the reflexions of the origin $x_1, x_2, \cdots, x_n$ and take for their equations in circular coördinates:

$$xt_\alpha + y = x_\alpha t_\alpha,$$

where $(\alpha = 1, 2, \cdots, n)$

$$x_\alpha t_\alpha = y_\alpha.$$

§ 2. *The center-circle and characteristic constants of an n-line.*

The two lines:

$$xt_1 + y = x_1 t_1,$$

$$xt_2 + y = x_2 t_2$$

meet at the point:

$$x_{12} = \frac{x_1 t_1}{t_1 - t_2} + \frac{x_2 t_2}{t_2 - t_1}.$$

Consider three lines and let

$$x = \frac{x_1 t_1 (t_1 - t)}{(t_1 - t_2)(t_1 - t_3)} + \frac{x_2 t_2 (t_2 - t)}{(t_2 - t_3)(t_2 - t_1)} + \frac{x_3 t_3 (t_3 - t)}{(t_3 - t_1)(t_3 - t_2)}.$$

This is of the form:

$$x = a_1 - a_2 t,$$

and therefore is a circle whose center is a_1 and radius is $|a_2|$. But $x = x_{12}$ when $t = t_3$, and similarly the circle passes through the intersection of each two of the three lines. Thus the circle is the circumcircle of the 3-line, of which the circumcenter is

$$a_1 = \sum \frac{x_1 t_1^2}{(t_1 - t_2)(t_1 - t_3)},$$

and the radius is $|a_2|$ where

$$a_2 = \sum \frac{x_1 t_1}{(t_1 - t_2)(t_1 - t_3)}.$$

Consider 4-lines and let

$$x = \sum \frac{x_1 t_1^2 (t_1 - t)}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)}.$$

Then x moves on a circle; and, when $t = t_4$, x is the circumcenter just found. But similarly when $t = t_1$, t_2, or t_3, x is a circumcenter of a 3-line selected from the 4-line. Thus we have STEINER'S theorem that the circumcenters of the four 3-lines selected from a 4-line lie on a circle. The center of STEINER'S circle is

$$a_1 = \sum \frac{x_1 t_1^3}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)};$$

the radius is $|a_2|$ where

$$a_2 = \sum \frac{x_1 t_1^2}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)}.$$

But it is now clear that the reasoning admits of endless repetition. Thus from a 5-line we have five 4-lines; the centers of their STEINER circles lie on a circle, a result given by KANTOR (*Über das vollständige Fünfseit*, Wiener Berichte, vol. 78, p. 167). From a 6-line we have six 5-lines; *the centers of their Kantor circles lie on a circle.* Call this the *center-circle* of the 6-line. From a 7-line we have seven 6-lines; *the centers of their center-circles lie on a circle. And so on ad infinitum.*

For an n-line, when $n > 2$, *the center-circle is given by*

$$x = a_1 - a_2 t,$$

where

$$a_1 = \sum \frac{x_1 t_1^{n-1}}{(t_1 - t_2) \cdots (t_1 - t_n)},$$

$$a_2 = \sum \frac{x_1 t_1^{n-2}}{(t_1 - t_2) \cdots (t_1 - t_n)}.$$

Generally * we take, as *characteristic constants of an n-line*,

* For these characteristic constants of an n-line a double notation might be used, such as a_α^n, but it seems better in practice mentally to supply the number of lines we wish to consider, and to write merely a_α.

$$a_a = \sum \frac{x_1 t_1^{n-a}}{(t_1 - t_2)\cdots(t_1 - t_n)}, \qquad (a = 1, 2, \cdots, n). \tag{4}$$

The conjugate of a_a is b_a where

$$b_a = \sum \frac{y_1 t_1^{a-n}}{\left(\frac{1}{t_1} - \frac{1}{t_2}\right)\cdots\left(\frac{1}{t_1} - \frac{1}{t_n}\right)}$$

$$= (-)^{n-1} t_1 t_2 \cdots t_n \sum \frac{x_1 t_1 \cdot t_1^{n-2} \cdot t_1^{a-n}}{(t_1 - t_2)\cdots(t_1 - t_n)}$$

$$= (-)^{n-1} t_1 t_2 \cdots t_n \cdot a_{n+1-a},$$

that is, if one sets

$$s_n = t_1 t_2 \cdots t_n,$$

$$b_a = (-)^{n-1} s_n a_{n+1-a}. \tag{5}$$

The n equations (4) can be solved at once for $x_1 \cdots x_n$. Thus

$$x_1 = a_1 - a_2\sigma_1 + a_3\sigma_2 + \cdots + (-)^{n-1} a_n \sigma_{n-1},$$

where σ_m is the sum of products m at a time of $t_2, t_3, \cdots, t_n$.

§3. *The node of an n-line.*

When we omit the line x_1 from the n-line we have an $(n-1)$-line whose constants are

$$a_1 - a_2 t_1, \quad a_2 - a_3 t_1, \cdots, a_{n-1} - a_n t_1.$$

When we omit two lines x_1, x_2 from the n-line we have an $(n-2)$-line whose constants are

$$a_1 - a_2(t_1 + t_2) + a_3 t_1 t_2, \quad a_2 - a_3(t_1 + t_2) + a_4 t_1 t_2, \cdots.$$

And so on.

We consider now the equation:

$$x = a_1 - a_2(t_1 + t_2) + a_3 t_1 t_2.$$

When we replace t_2 by a variable t we have the center-circle of the $(n-1)$-line obtained by omitting t_1. Thus if t_1 denote any assigned turn all the center-circles of the $(n-1)$-lines formed from an n-line are included in

$$x = a_1 - a_2(t_1 + t) + a_3 t_1 t. \tag{6}$$

We now prove that all these circles meet at a point. For the case $a_2 = 0$, the circles are all the same circle.

For the case $a_2 \neq 0$ suppose x given. The equation

$$a_1 - x - a_2(z_1 + z) + a_3 z_1 z = 0$$

is that of a complex involution I_1^2, wherein to the unit circle on which t moves corresponds another circle. The involution pairs off the circles in the plane and it is convenient to speak of the paired circles as *partners*. But circles can be their own partners, when, namely, they go either through the double points or about the double points (i. e., when the double points are inverse points of the circle). It is convenient to speak of these as circles of the involution, or as double circles.

When the unit circle is not a circle of the involution there are two cases: *either* the unit circle cuts its partner and then the intersections are a pair of points of the I_1^2, and the only pair on the unit circle, *or* the unit circle does not cut its partner and then the common inverse points of the two belong to I_1^2.

But when the unit circle is a circle of I_1^2, then there are ∞ pairs of I_1^2 on it forming a common projective involution.

All this is verified at once by means of the canonical form of I_1^2:

$$z + z' = 0\,.$$

When a relation involving turns, say

$$f(t_1, t_2 \cdots \mid a_1, a_2, \cdots) = 0,$$

is the same, save as to a factor, as the conjugate relation:

$$f(1/t_1, 1/t_2, \cdots \mid b_1, b_2, \cdots) = 0\,,$$

we shall say that the relation is *self-conjugate*.

Now in our case in order that when z_1 is any point t_1 on the unit circle the partner z shall also be on the unit circle we require that the above equation (6) shall be self-conjugate. That is

$$x = a_1 - a_2(t_1 + t) + a_3 t_1 t$$

and

$$y = b_1 - b_2(1/t_1 + 1/t) + b_3/t_1 t$$

are to be not two equations determining t_1 and t but merely one equation. And this is the case when

$$x = a_1 - a_2 b_3/b_2;$$

for then the equation (6) is

$$\frac{b_3}{b_2} - (t_1 + t) + \frac{a_3}{a_2} t_1 t = 0\,,$$

which is manifestly self-conjugate. Whatever turn t_1 may be, a turn t can be found satisfying this equation, and the corresponding value of x,

$$x = a_1 - a_2 b_3 / b_2, \tag{7}$$

is thus common to all the circles. Hence the theorem:

All the center-circles of the $(n-1)$-lines formed from an n-line meet at a point.

It is natural to consider the equation (6) in connection with the equation:

$$x = a_1 - 2a_2 t + a_3 t^2, \tag{8}$$

from which (6) is deduced by polarizing. This is the map-equation of a limaçon, *the limaçon of the n-line.*

What is the relation of the circles to the limaçon?

First. The t_1-circle and the limaçon have in common the point for which $t = t_1$.

Second. They touch at that point. For at t_1 one has

$$D_t x = -a_2 + a_3 t_1 \text{ for the circle,}$$
$$D_t x = -2a_2 + 2a_3 t_1 \text{ for the limaçon;}$$

whence the direction at the point is the same in both cases.

Third. The point where all the circles meet is the node of the limaçon. For when $x = a_1 - a_2 b_3 / b_2$ we have

$$\frac{b_3}{b_2} - 2t + \frac{a_3}{a_2} t^2 = 0;$$

so that we have the same point of the limaçon for two different values of t.

We may speak of this point as the *node of the n-line.*

A name is necessary for the system of circles, given by (6); that is, for the general system of circles passing through a point and having their centers on a circle.

I shall call them *penosculants** of the limaçon:

$$x = a_1 - 2a_2 t + a_3 t^2.$$

Generally when x is a rational integral algebraic function of t, and the polynomial is polarized once, or more, the resulting curves may be termed penosculants.

Thus the limaçons of the $(n-1)$-lines formed from an n-line are included in

$$x = a_1 - a_2(t_1 + 2t) + a_3(2t_1 t + t^2) - a_4 t_1 t^2;$$

* The name may serve to suggest the analogy as to formation with the theory of osculants of rational curves developed primarily by STUDY, Leipziger Berichte, 1886. The name *oscuant* was introduced by JOLLES in his Habilitationsschrift, Aachen, 1886.

and these are first penosculants of the curve:

$$x = a_1 - 3a_2t + 3a_3t^2 - a_4t^3 .$$

So the center-circles of all $(n-2)$-lines formed from the n-line are included in

$$x = a_1 - a_2s_1 + a_3s_2 - a_4s_3 ,$$

where s_α is a sum of products of three turns, and these are second penosculants of the same curve.

We shall not examine this curve now, but shall rather seek to show by taking more familiar curves that there is a place for the theory of penosculants. But before leaving the method of the a's, we shall apply it to CLIFFORD'S theorem.

§ 4. *Clifford's chain.* *

From the point of view of this method CLIFFORD'S chainwise extension of MIQUEL'S theorem is as follows:

The circumcircle of a three-line is:

$$x = a_1 - a_2t .$$

The four such circles of a 4-line are included in

$$x = a_1 - a_2(t_1 + t) + a_3t_1t ;$$

and meet at the point (7),

$$x = a_1 - a_2b_3/b_2 ,$$

that is, since from (5) $b_3/b_2 = a_2/a_3$ for a 4-line, at the point

$$x = a_1 - a_2^2/a_3 ,$$

the two turns t_1, t for this x being the one a_2/a_3 and the other arbitrary.

The five such points of a 5-line are included in

$$x = a_1 - a_2s_1 + a_3s_2 - a_4s_3 ,$$

where s_1, s_2, s_3 are the sums of products (one, two, three at a time) of three turns.

Now consider this complex involution, regarding x as given. There is in an involution I_1^3 a neutral pair, that is a pair whose third element is arbitrary. Here the neutral pair is given by

$$(9) \qquad \begin{cases} x = a_1 - a_2(t_1 + t) + a_3t_1t, \\ 0 = a_2 - a_3(t_1 + t) + a_4t_1t. \end{cases}$$

* CLIFFORD'S memoir (loc. cit.) was published in 1870; KANTOR proved the same theorem in 1876, and subsequently published a second proof which is practically the same as CLIFFORD'S.

But for five lines the last equation is self-conjugate. Thus for any x the neutral pair is on the unit circle; whatever turn t_1 may be, the turn t can be found to satisfy the last equation; and the point x is then on the circle:

$$\begin{vmatrix} a_1 - x & a_2 \\ a_2 & a_3 \end{vmatrix} = t \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix},$$

found by eliminating $t_1 + t$ and writing t for $t_1 t$. This then is MIQUEL'S circle.

For six lines the two equations (9) become:

$$(10) \qquad \begin{cases} x = a_1 - a_2 s_1 + a_3 s_2 - a_4 s_3, \\ 0 = a_2 - a_3 s_1 + a_4 s_2 - a_5 s_3, \end{cases}$$

where the s_1, s_2, s_3 are for three turns. Now for six lines the last equation is self-conjugate. The equations determining the neutral pair of this involution I_1^3 are conjugate, namely:

$$a_2 - a_3 s_1 + a_4 s_2 = 0, \quad a_3 - a_4 s_1 + a_5 s_2 = 0,$$

where s_n is a sum of products of two turns. Hence the neutral pair is on the unit circle. The equations of all MIQUEL circles hold when

$$\begin{aligned} x &= a_1 - a_2 s_1 + a_3 s_2, \\ 0 &= a_2 - a_3 s_1 + a_4 s_2, \\ 0 &= a_3 - a_4 s_1 + a_5 s_2; \end{aligned}$$

that is, all MIQUEL circles meet at the CLIFFORD point:

$$\begin{vmatrix} a_1 - x & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 0.$$

For seven lines the equations (10) determining the CLIFFORD point become:

$$(11) \qquad \begin{cases} x = a_1 - a_2 s_1 + a_3 s_2 - a_4 s_3, \\ 0 = a_2 - a_3 s_1 + a_4 s_2 - a_5 s_3, \\ 0 = a_3 - a_4 s_1 + a_5 s_2 - a_6 s_3; \end{cases}$$

where the last two are conjugate. Eliminating s_1 and s_2 we see that the seven CLIFFORD points lie on the CLIFFORD circle:

$$\begin{vmatrix} a_1 - x & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = t \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \end{vmatrix}.$$

It is now clear that the argument is general and we can say: *the Clifford point for* $2p$ *lines is given by*

$$\begin{vmatrix} a_1 - x & a_2 & \cdots & a_p \\ a_2 & a_3 & \cdots & a_{p+1} \\ \vdots & & & \\ a_p & a_{p+1} & \cdots & a_{2p-1} \end{vmatrix} = 0;$$

and the equation of the Clifford circle for $2p+1$ *lines is given by*

$$\begin{vmatrix} a_1 - x & a_2 & \cdots & a_p \\ a_2 & a_3 & \cdots & a_{p+1} \\ \vdots & & & \\ a_p & a_{p+1} & \cdots & a_{2p-1} \end{vmatrix} = t \begin{vmatrix} a_2 & a_3 & \cdots & a_{p+1} \\ a_3 & a_4 & \cdots & a_{p+2} \\ \vdots & & & \\ a_{p+1} & a_{p+2} & \cdots & a_{2p} \end{vmatrix}.$$

We have thus the coördinates of Clifford's chain.

The CLIFFORD circle ceases to exist if

$$\begin{vmatrix} a_3 & a_4 & \cdots & a_{p+1} \\ a_4 & a_5 & \cdots & a_{p+2} \\ \vdots & & & \\ a_{p+1} & a_{p+2} & \cdots & a_{2p-1} \end{vmatrix} = 0,$$

since then s_p is a constant. But if no further conditions are imposed, we can solve the equations of the type (9) or (11) in the form:

$$x = A_0 + A_1 s_1,$$

and

$$x = A_p + A_{p-1} s_{p-1}$$

or

$$y = B_p + B_{p-1} s_1/s_p,$$

whence on eliminating s_1 we have a *line*.

We turn now to the consideration of some penosculants.

§ 5. *Penosculants of the limaçon.*

The theory of five lines appears in its simplest form when considered in connection with the limaçon. Starting with this curve we write it

$$x = 2at + \beta t^2,$$

where a and β are real.

Here the center is the origin, the node is $-\beta$, the focus is $-a^2/\beta$. A penosculant (say P_1) is

$$x = a(t_1 + t) + \beta t_1 t.$$

It appears from the (9) of the last section that the involution:

$$a + \beta(t_1 + t) + at_1t = 0$$

is of importance.*

Eliminating $t_1 + t$ we have

$$\begin{vmatrix} x & a \\ a & -\beta \end{vmatrix} = t_1 t \begin{vmatrix} a & -\beta \\ -\beta & a \end{vmatrix},$$

a circle with center at the focus, and passing through the node. We can call this the circle of inversion of the limaçon, since by an inversion in this circle the limaçon passes into itself.

Hence the point:

$$x = a(t_1 + t) + \beta t_1 t,$$
$$0 = a + \beta(t_1 + t) + at_1 t,$$

is the point where P_1 meets the circle of inversion (apart from the node).

Two penosculants P_1 and P_2 meet at

$$x_{12} = a(t_1 + t_2) + \beta t_1 t_2.$$

Consider now the expression:

$$x = a(t_1 + t_2 + t) + \beta(t_1t_2 + t_1t + t_2t) + at_1t_2t. \tag{12}$$

It is a circle with center x_{12}, and it passes through the points where P_1 and P_2 meet the circle of inversion. That is:

If two penosculants P_1, P_2 meet at x_{12} and meet the circle of inversion at m_1, m_2, the node where all penosculants meet being disregarded, then a circle with center x_{12} will pass through m_1 and m_2.

From the symmetry of (12) *the three such circles for three penosculants meet at a point, namely:*

$$x = a(t_1 + t_2 + t_3) + \beta(t_1t_2 + t_1t_3 + t_2t_3) + at_1t_2t_3,$$

or say

$$x = as_1 + \beta s_2 + as_3, \quad (s \text{ for 3 turns}).$$

The conjugate equation is

$$ys_3 = as_2 + \beta s_1 + a.$$

Multiply by t_4 and add. Then

$$x + ys_4 = as_1 + \beta s_2 + as_3, \quad (s \text{ for 4 turns}). \tag{13}$$

Hence: *For four penosculants the four points lie on a line.*

Naturally then five penosculants will give five lines and thus we get the MIQUEL figure as completed by KANTOR.

* It is verified at once that points of the limaçon whose parameters are pairs of the involution are the ends of a focal chord.

But, in fact, the theorem proved for four penosculants is the key to the theory of five lines.

From the standpoint of geometric drawing also this reverse view is the simplest. For the tedious determination of circles from three points is altogether avoided; the centers are given.

Still considering the limaçon:

$$x = 2at + \beta t^2,$$

we determine as follows the circle through the intersections (other than the node) of three penosculants.

We have for the intersection of two penosculants,

$$x_{12} = a(t_1 + t_2) + \beta t_1 t_2,$$

or

$$\beta x_{12} + a^2 = (a + \beta t_1)(a + \beta t_2) = z_1 z_2 \text{ say.}$$

Hence the three points x_{12}, x_{13}, x_{23} lie on the circle:

$$\beta x + a^2 = z_1 z_2 z_3 / z,$$

where

$$z = a + \beta t.$$

In this mapping the focus $-a^2/\beta$ corresponds to $t = \infty$, the point $x = \infty$ to $t = -a/\beta$. The center of the circle or inverse of $x = \infty$ corresponds to the inverse of $-a/\beta$ as to the unit circle, or to $-\beta/a$, and is therefore

$$\beta x + a^2 = \frac{a}{a^2 - \beta^2} z_1 z_2 z_3.$$

The centers of the four such circles, for four penosculants taken by threes, lie on the circle:

$$\beta x + a^2 = \frac{a}{a^2 - \beta^2} \frac{z_1 z_2 z_3 z_4}{z};$$

and the center of this circle is as before

$$\beta x + a^2 = \left(\frac{a}{a^2 - \beta^2}\right)^2 z_1 z_2 z_3 z_4.$$

We have then again a geometric chain proceeding *ad infinitum*. Thus the center-circle theorem proved in § 2 for lines is true also for penosculants of a limaçon; or explicitly:

Consider the circles whose centers lie on a given circle and which pass through a given point. Three of the circles have a circumcircle, or circle through their free intersections.

Four circles have four circumcircles; their centers lie on a circle. Call this the center-circle of the four.

Five circles have five center-circles ; their centers lie on a circle. And so on.

This theorem is the direct continuation of the theory of § 3, and is parallel to that of § 2. But there is also a parallel of § 3.

For the center-circle of $n-1$ penosculant circles is

$$\beta x + a^2 = \left(\frac{a}{a^2-\beta^2}\right)^{n-3} \frac{z_1 z_2 \cdots z_{n-1}}{z}.$$

Now

$$z z_n = (a+\beta t)(a+\beta t_n).$$

Hence if

$$z z_n = a^2 - \beta^2,$$

then

$$\beta + a(t+t_n) + \beta t t_n = 0,$$

a self-conjugate equation. Hence the center-circle passes through the point:

$$\beta x + a^2 = \frac{a^{n-3}}{(a^2-\beta^2)^{n-2}} z_1 z_2 \cdots z_n. \tag{14}$$

That is, *when we have n circles whose centers lie on a circle and which pass through a point, if we take the center-circle of each set of n — 1 circles, these n center-circles meet at a point.*

That CLIFFORD'S chain exists also for the penosculants of a limaçon is evident, for CLIFFORD'S theorem is by inversion true when the system of lines is replaced by *any* system of circles through a point.

§ 6. *The quadratic involution arising from 2p lines of a conic.*

When we invert the limaçon whose penosculants we have been considering into a conic the penosculants become the lines of the conic. We have theorems, which it is superfluous to state, in which the inverse of a focus of the conic as to a circle replaces the center of the circle. But some geometric statements will be facilitated by turning now to the conic, which we take to be an ellipse.

We write the ellipse

$$2x = t + \mu^2/t,$$

where μ is real.

The center is the origin, the foci are $\pm\mu$, the axes are $|1 \pm \mu^2|$.

The tangent at t_1, i. e., the line t_1 of the ellipse, is

$$x = \frac{t_1 t + \mu^2}{t_1 + t}. \tag{15}$$

For this is a line since $x = \infty$ when $t = -t_1$; the line passes through the point t_1 of the ellipse; and it touches the ellipse at this point since when $t = t_1$

$$D_t x = \frac{t_1^2 - \mu^2}{4t_1^2} \text{ for the line,}$$

$$D_t x = \frac{t_1^2 - \mu^2}{2t_1^2} \text{ for the ellipse;}$$

whence the direction of motion is the same in both cases.

From the mode of formation we may call this line a first penosculant of the ellipse.

The intersection of two such lines is the second penosculant, or penosculant point:

$$x_{12} = \frac{t_1 t_2 + \mu^2}{t_1 + t_2}.$$

We write (15) in the form:

$$\frac{x+\mu}{x-\mu} = \frac{t_1 + \mu}{t_1 - \mu}\frac{t+\mu}{t-\mu},$$

or

$$\frac{x+\mu}{x-\mu} = z_1 z,$$

where

$$z = \frac{t+\mu}{t-\mu}.$$

Then

$$\frac{x+\mu}{x-\mu} = \frac{z_1 z_2 z_3}{z} \tag{16}$$

is a circle; and, when $t = t_3$, $x = x_{12}$. Thus the circle is the circumcircle of the three lines of the ellipse.

When $t = -t_4$, then

$$z = \frac{-t_4 + \mu}{-t_4 - \mu} = \frac{1}{z_4}.$$

Hence the four circumcircles of four lines of the ellipse meet at the point:

$$\frac{x+\mu}{x-\mu} = z_1 z_2 z_3 z_4.$$

For five lines the five such points lie on the circle:

$$\frac{x+\mu}{x-\mu} = \frac{z_1 z_2 z_3 z_4 z_5}{z},$$

the MIQUEL circle of the five lines. Evidently then:

The Clifford point of $2p$ *lines of an ellipse is*

$$\frac{x+\mu}{x-\mu} = z_1 z_2 \cdots z_{2p}. \tag{17}$$

The Clifford circle of $2p-1$ *lines is obtained by writing* $1/z$ *for* z_{2p}.

The inverse of the focus μ as to the circumcircle (16) is found as follows: One has the corresponding values:

$$x = \mu, \quad z = 0, \quad t = -\mu.$$

The inverse of $-\mu$ as to the unit circle is $-1/\mu$; therefore the inverse of μ as to the circumcircle is given by:

$$\frac{x+\mu}{x-\mu} = \frac{1+\mu^2}{1-\mu^2} z_1 z_2 z_3.$$

For four lines the four such points lie on the circle:

$$\frac{x+\mu}{x-\mu} = \frac{1+\mu^2}{1-\mu^2} \frac{z_1 z_2 z_3 z_4}{z},$$

and the inverse of the focus μ as to this is as before

$$\frac{x+\mu}{x-\mu} = \left(\frac{1+\mu^2}{1-\mu^2}\right)^2 z_1 z_2 z_3 z_4.$$

Thus working with the one focus we have (for the first chain of § 5 transferred to the ellipse) the series of points

$$\frac{x+\mu}{x-\mu} = \left(\frac{1+\mu^2}{1-\mu^2}\right)^{n-2} \cdot z_1 z_2 \cdots z_n;$$

and in the same way for the other focus we have

$$\frac{x'+\mu}{x'-\mu} = \left(\frac{1-\mu^2}{1+\mu^2}\right)^{n-2} \cdot z_1 z_2 \cdots z_n.$$

The inverse of the focus μ as to a CLIFFORD circle is similarly, from (17),

$$\frac{x+\mu}{x-\mu} = \frac{1+\mu^2}{1-\mu^2} z_1 z_2 \cdots z_{2p-1};$$

that of the other focus $-\mu$ is

$$\frac{x+\mu}{x-\mu} = \frac{1-\mu^2}{1+\mu^2} z_1 z_2 \cdots z_{2p-1}.$$

If in the preceding process we take the foci alternately it is clear that, for any $2p$ lines of the conic, we arrive at the CLIFFORD point (17):

$$\frac{x+\mu}{x-\mu} = z_1 z_2 \cdots z_{2p},$$

inasmuch as each factor $(1+\mu^2)/(1-\mu^2)$ introduced by using the positive focus μ is cancelled by the factor $(1-\mu^2)/(1+\mu^2)$ introduced by the other focus. It is clear also that we have entirely commutative geometric constructions; we

arrive at this same point in whatever order we employ the foci, provided we use each $p-1$ times.

Thus for 6 lines we may take the inverses of μ as to all circumcircles; we get twenty points lying by fours on 15 circles—one for each 4-line; then the inverses of $-\mu$ as to these 15 circles, getting 15 points lying on 6 circles, 5 on each circle; then the inverses of μ as to these 6 circles, getting 6 points on a circle; and finally the inverse of $-\mu$ as to this circle. But the order in which the foci are taken is indifferent; the point arrived at is always:

$$\frac{x+\mu}{x-\mu} = z_1 z_2 z_3 z_4 z_5 z_6.$$

Consider the complex involution I_1^2 defined by

$$\frac{x+\mu}{x-\mu}\cdot\frac{x'+\mu}{x'-\mu} = z_1 z_2 \cdots z_{2p}. \tag{18}$$

This involution has the following properties:

(i) *Its center is the Clifford point of the* $2p$ *lines.*

(ii) *The foci are a pair of the involution* I_1^2.

(iii) *The Clifford point of* $2q$ *lines and that of the remaining* $2(p-q)$ *lines are a pair of* I_1^2. The Clifford point of two lines means merely their intersection.

(iv) *The Clifford circle of* $2q-1$ *lines and that of the remaining lines are partners.* The Clifford circle of a line is merely the line itself.

As to proof, (iii) is obtained by breaking up (18) into

$$\frac{x+\mu}{x-\mu} = z_1 z_2 \cdots z_{2q},$$

$$\frac{x'+\mu}{x'-\mu} = z_{2q+1} \cdots z_{2p};$$

and (iv) follows from (iii) by considering that Clifford points lie on Clifford circles.

The circle of similitude of two circles is the locus of centers of involutions in which the two are partners. For any line through the center of an I_1^2 passes into a line through the center; hence the tangents from the center to a circle pass into the tangents from the center to the partner circle; whence by the principle of isogonality the partner circles subtend equal angles at the center. Hence:

Given $2p$ *lines of an ellipse, the circle of similitude of the Clifford circles of any* $2q-1$ *lines and of the remaining lines passes through the Clifford point of all the lines.*

We might also consider higher involutions, for instance the I_2^3:

$$\frac{x+\mu}{x-\mu}\cdot\frac{x'+\mu}{x'-\mu}\cdot\frac{x''+\mu}{x''-\mu}=z_1z_2\cdots z_n: \tag{19}$$

Thus for 6 lines of an ellipse we infer that all the triads of points obtained by dividing the 6 lines into three pairs belong to an I_2^3 of which the foci are the neutral pair.

§ 7. *Penosculants of a hypocycloid.*

The penosculants were defined in § 3 for *integral* algebraic functions. But, as we have seen in the case of the ellipse, the notion may be advantageously used also when x is a rational *fractional* algebraic function of t. We shall take an example of this, which is the key to an interesting generalization.

We consider the hypocycloid

$$(n-1)\,x=nt-1/t^n.$$

The equation

$$(n-1)\,x=t_1+(n-1)\,t-1/t_1t^{n-1}$$

defines a first penosculant P. It is a hypocycloid touching the given one at t_1. It has cusps when

$$D_tx=0,$$

or when

$$t_1t^n+1=0,$$

or when

$$(n-1)x=nt-1/t^n.$$

Thus the cusps of a first penosculant are on the given hypocycloid. *

A second penosculant is defined by

$$(n-1)\,x=t_1+t_2+(n-2)\,t-1/t_1t_2t^{n-2}.$$

It is the first penosculant at t_1 of the first penosculant at t_2; or it is the first penosculant at t_2 of the first penosculant at t_1.

And so on till we reach the completely polarized equation:

$$(n-1)\,x=s_1-1/s_n, \qquad (s \text{ for } n \text{ turns}); \tag{20}$$

when one of the n t's is variable this equation is the equation of a segment of a line or penosculant line, and when all t's are fixed it defines a penosculant point, in which meet all the penosculant lines formed from $n-1$ of the n t's.

An independent meaning of this point will be proved later. First we develop its more obvious properties.

(i) Take $2n$ points of the curve; divide them into any two sets of n each; all the centroids of the penosculant points of such two complementary sets lie on a line. For if

* See a memoir in the American Journal of Mathematics, vol. 16, p. 196.

$$(n-1)x = t_1 + \cdots + t_n - 1/t_1 \cdots t_n,$$

and

$$(n-1)x' = t_{n+1} + \cdots + t_{2n} - 1/t_{n+1} \cdots t_{2n},$$

then

$$(n-1)(x+x') = s_1 + t + 1/ts_{2n}, \qquad (s \text{ for } 2n \text{ turns}),$$

where $-1/t = t_1 t_2 \cdots t_n$. But this is the map-equation of a segment of a line; hence all the points $(x+x')/2$ lie on a line. The line is a penosculant line of

$$2(n-1)x = (2n+1)t + 1/t^{2n+1}.$$

(ii) Take $n+1$ points of the curve; the penosculant points of the various n points are included in

$$(n-1)x = s_1 - t - t/s_{n+1}, \qquad (s \text{ for } n+1 \text{ turns}). \tag{21}$$

That is, *the $n+1$ penosculant points lie on a circle.*

(iii) Taking $n+2$ points of the curve we have $n+2$ such circles; writing

$$t = -t_1 t_2 \cdots t_{n+2},$$

we see that

$$x = t_1 + t_2 + \cdots + t_{n+2} + t_1 t_2 \cdots t_{n+2}.$$

Hence *the $n+2$ circles meet in the point:*

$$x = s_1 + s_{n+2}, \qquad (s \text{ for } n+2 \text{ turns}). \tag{22}$$

(iv) Taking $n+3$ points of the curve we have $n+3$ such points, included in

$$x = s_1 - t + s_{n+3}/t, \qquad (s \text{ for } n+3 \text{ turns}). \tag{23}$$

This last is the map-equation of a segment; hence *the $n+3$ points lie on a line.*

Take $n+4$ points of the curve; the $n+4$ lines are included in

$$x = s_1 - t - t' + s_{n+4}/tt', \qquad (s \text{ for } n+4 \text{ turns}), \tag{24}$$

and are therefore tangents of a hypocycloid of the third class.

For $n+5$ points of the curve the hypocycloids are penosculants of an asteroid, or hypocycloid of the fourth class. And so on.

When we come to $2n+2$ points of the curve, say $t_1, t_2, \cdots, t_{2n+2}$, we have again a hypocycloid which is merely the original one displaced. And there is this remarkable reciprocity, that if we take any n of the points $t_1, t_2, \cdots, t_{2n+2}$ of this second curve and repeat the process, selecting always from the remaining points, we recover the original curve. This reciprocity, for the case $n=2$, was found in another way by KANTOR.*

* *Die Tangentengeometrie an der Steiner'schen Hypocycloide*, Wiener Berichte, vol. 78, p. 232.

In this case of $n = 2$ the meaning of the penosculant point is evident. The equation (20) is

$$x = t_1 + t_2 - 1/t_1 t_2 .$$

For a variable t_2 this is the tangent at t_1 (or rather that segment of the tangent which is within the curve). Hence the point is the intersection of the tangents at t_1, t_2.

How to attach a meaning to the penosculant point (20) in general may be inferred from the case $n = 4$.

The line-equation of the hypocycloid

$$3x = 4t - 1/t^4$$

is

$$xt^4 + yt = t^5 + 1 .$$

Consider also the curve whose line-equation is

$$xt^3 + y = x_0 t^3 + y_0 + at^2 + bt . \tag{25}$$

The lines for which

$$t^5 + 1 = x_0 t^4 + y_0 t + at^3 + bt^2$$

are common tangents to the two curves. Hence for five turns with $s_5 = -1$ if

$$x_0 = s_1 , \quad y_0 = -s_4 , \quad a = -s_2 , \quad b = s_3 ,$$

these being pairs of conjugates by virtue of the relation $s_5 = -1$, a curve (25) is determined which touches the 4 lines t_1, t_2, t_3, t_4 of the hypocycloid.

The line-equation of the curve is

$$(x - s_1)t^3 + y + s_4 = -s_2 t^2 + s_3 t ;$$

and its point-equation is

$$3(x - s_1) = -2s_2/t + s_3/t^2 .$$

The curve is therefore a cardioid, and its center is

$$x = s_1 = t_1 + t_2 + t_3 + t_4 - 1/t_1 t_2 t_3 t_4 ,$$

the point sought.

Thus the penosculant point of 4 points of the curve

$$3x = 4t - 1/t^4$$

is the center of the (unique) cardioid which touches the 4 tangents.

It will be noticed that we are dealing with *curves of direction.** The hypocycloid assigns a definite direction to its tangents, and these four directed lines determine the cardioid. There would be 2^3 cardioids touching four lines not directed.

---

* LAGUERRE, Comptes Rendus, 1865.

The use of this determination of the penosculant point lies in the following parallel. This memoir handles mainly the intersections of n lines, that is, the lines are primarily taken 2 at a time. There is a specially simple case, namely, when the lines all touch a hypocycloid of class three. From this case we have passed now to other simple cases: when all the lines touch other hypocycloids. But this gives the key to the general case of any n lines; if we (continuing our example) group the lines 4 at a time we are to consider the cardioids which touch the lines by fours. To the centers of these cardioids the general theorems of the present memoir will equally apply; thus for 5 directed lines the centers of the cardioids will lie on a circle, and so on. In fact the algebra will be the same, the meaning of certain constants alone being altered.

To see this, we observe that a tangent of a cardioid whose center is x_0 is

$$xt^3 + y = x_0t^3 + y_0 + at^2 + bt,$$

or, if p be the distance from the origin to this line,

$$x_0t^3 + y_0 + at^2 + bt = 2pt^{\frac{3}{2}}.$$

Hence for four given tangents

$$x_0 = 2 \begin{vmatrix} p_1t_1^{\frac{3}{2}} & t_1^2 & t_1 & 1 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ p_4t_4^{\frac{3}{2}} & t_4^2 & t_4 & 1 \end{vmatrix} \div \begin{vmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ t_4^3 & t_4^2 & t_4 & 1 \end{vmatrix}.$$

If then we write x_1 for $2p_1t_1^{\frac{3}{2}}$ we have:

$$x_0 = \sum \frac{x_1}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)},$$

an expression of the form used in § 2 and in fact that there denoted, for a 4-line, by a_3.

Evidently then in the argument based on the constants a we can substitute the center of the cardioid touching four directed lines for the intersection of two lines. And so in general for the intersection of two lines we may substitute the center x_0 of a curve touching $n+1$ directed lines, the line-equation of this curve being

$$xt^n + y = x_0t^n + y_0 + at^{n-1} + bt + a't^{n-2} + b't^2 + \cdots.$$

HAVERFORD COLLEGE,
January, 1900.

ON RELATIVE MOTION*

BY

ALEXANDER S. CHESSIN

PART I.

Contents.

CHAPTER		PAGE
	INTRODUCTION	116
I.	—Differential equations of Lagrange for relative motion	118
II.	—Canonical equations of Hamilton-Jacobi for relative motion	122
III.	—Convection as a perturbation. The perturbative function of convective motion	124
IV.	—Digression on some transformation-formulas in passing from cartesian coördinates to a system of independent coördinate parameters	127
V.	—Application of the formulas of the preceding chapter to the perturbative function of convective motion	132
VI.	—Another digression on transformations of coördinates in the equations of dynamics	136
VII.	—Return to the perturbative function of convective motion	141
VIII.	—Decomposition of the kinetic energy of a material system. Mechanical significance of the function G_2	143
IX.	—Decomposition of the kinetic energy of a solid body. Mechanical interpretation of the function G_2	153
X.	—Other methods for obtaining the differential equations of relative motion in Lagrange's second form or in the canonical form of Hamilton-Jacobi	162
XI.	—Some remarks concerning the differential equations of Lagrange and of Hamilton-Jacobi	166

INTRODUCTION.

The theory of relative motion as developed in this paper originated in a memoir by E. BOUR in 1863.† It deals mainly with the so-called "second form" of differential equations of LAGRANGE and with the canonical system of differential equations of HAMILTON-JACOBI. BOUR, very modestly, claimed no dis-

* Presented to the Society at the Columbus meeting, August 25, 1899. Received for publication September 23, 1899.

† Journal de Liouville, ser. 2, vol. 8.

covery. "There is no theory of relative motion, properly speaking," he remarks in the introduction to his memoir, and, in a certain sense, this is true as I have shown in the course of this paper (See chap. X, §1). Nevertheless, BOUR's memoir was followed by several important applications, and the differential equations established by him for relative motion have been used since with great advantage. In this connection the memoir: *Sur l'application de la méthode de Lagrange à divers problèmes de mouvement relatif*,* by PH. GILBERT, and the dissertation: *Over de bettrekkelijke Beweging* † by KAMERLINGH-ONNES require special mention. The first is given over entirely to applications of LAGRANGE's equations, as the title clearly indicates. The second is an application of the canonical system of equations of HAMILTON-JACOBI. But KAMERLINGH-ONNES' paper is not restricted to applications. The author first generalizes BOUR's equations by assuming that the equations of constraint may involve the time explicitly. To KAMERLINGH-ONNES we also owe the first general expression of what I have called *the perturbative function of convective motion* in the subsequent pages.

The first part of my paper deals only with the *theory* of relative motion. The differential equations are derived from one fundamental principle embodied in the so-called "theorem of CORIOLIS." This enables us not only to write down the differential equations of relative motion *immediately* from the corresponding equations of absolute motion, but to obtain equations *as general as those known for absolute motion.* When the forces have a potential function my equations reduce to those of KAMERLINGH-ONNES. When, moreover, the constraints do not involve the time explicitly, they reduce to the equations of BOUR (Chapters I and II).

These equations are afterwards derived in a different way (Chapter X) and the connection between the two methods is brought forth (Chapter XI). At the same time SCHERING's generalized notion of a force function is obtained and it is shown how this generalization may serve the theory of relative motion.

In Chapter III the notion of perturbations is introduced into the theory of relative motion, convective motion (or, what is commonly known as "space motion") being the disturbing factor. The perturbative function so obtained is the sum of three terms, only two of which had been well defined by BOUR and KAMERLINGH-ONNES, namely, the functions K and L; the form and mechanical significance of the third term (denoted in this paper by G_2) remained indefinite if not unknown. For the function G_2 several expressions are given which show explicitly that its existence is due solely to the constraints of the material

*Annales de la Société scientifique de Bruxelles; 1882, pp. 270–373; 1883, pp. 11–110. Also in book form, Paris, ed. Gauthier-Villars. The book had two editions.

†Nieuw Archief voor Wiskunde (Amsterdam); vol. 5. This paper reproduces only the mathematical part of KAMERLINGH-ONNES' dissertation which deals with physical problems.

system (Chapters V and VII) and the mechanical significance of this function is established (Chapters VIII and IX).

Chapters IV and VI contain some necessary digressions into transformations of coördinates. The formulæ obtained in these two chapters are capable of wider applications.*

I should like to say a few words in regard to chapters VIII and IX. The manner in which the kinetic energy of a material system may be decomposed seems to me worthy of notice. Indeed, while it is quite obvious that the effect of constraints involving the time explicitly could be produced by submitting the given material system to a succession of different constraints not involving the time explicitly and equivalent to the given system of constraints at each particular moment; while, moreover, it is quite as obvious that the velocity w_i of a particle at a given moment will be the resultant of two velocities w_{i1} and w_{i2}, w_{i1} being that velocity which the particle would have if the time, as far as it enters explicitly in the equations of constraint, became fixed at that moment, and w_{i2} being the component velocity of the particle arising from the variation of the time, as far as it enters explicitly in the equations of constraint; yet, the fact that the kinetic energy of the material system $\frac{1}{2}\sum m_i w_i^2$ will be equal to the sum of the kinetic energies $\frac{1}{2}\sum m_i w_{i1}^2$ and $\frac{1}{2}\sum m_i w_{i2}^2$ is by no means obvious. I venture to say that this proposition may find more than one application in physics.

The second part of my paper which I hope to bring out in the near future will contain some applications of the theory expounded in Part I. Among the problems to be treated may be mentioned the intricate problem of FOUCAULT'S pendulum when the oscillations are not infinitely small, and the problem of FOUCAULT'S top which PH. GILBERT was unable to solve (loc. cit. § XX). Both these problems can be easily solved by means of the theory and formulas given in Part I, and they yield highly interesting results.

CHAPTER I.

DIFFERENTIAL EQUATIONS OF LAGRANGE FOR RELATIVE MOTION.

§1. *Preliminaries.* Let ΞHZ be a system of coördinate axes fixed in space and XYZ another system of axes moving with regard to the first. The motion of a material system referred to the axes ΞHZ will be called *absolute*; referred to the axes XYZ it will be called *relative*. The motion of the system XYZ itself with regard to the fixed axes ΞHZ will be called *convective motion*. I introduce this term in preference to the one commonly used "space motion." It

*See in this connection a paper by D. BOBYLEF in vol. 58 of the Memoirs (Zapiski) of the Imperial Academy of Sciences of St. Petersburg, 1888.

will be observed that the term here proposed comes nearer to the corresponding terms in French and in German: *mouvement d'entrainement* and *Führungsbewegung*. Sometimes the word *convection* will be used in place of convective motion.

Having broached the question of terminology I also propose to use the term *turning acceleration* in lieu of "compound acceleration." "Turning acceleration" is a translation of the term used by J. SOMOF in his treatise of mechanics and it owes its origin to its true mechanical significance. In his German translation of Somof's classical book Professor A. Ziwet uses the term "Rückkehrbeschleunigung,"* which may well be translated "turning acceleration."

§2. *Theorem of Coriolis.* I derive the differential equations for relative motion from the following principle:

Let (E) *be the differential equations of motion of a material system with regard to the invariable system* XYZ *on the supposition that convection did not exist, i. e., that the axes* XYZ *were fixed in space. To obtain the differential equations of relative motion of the material system with regard to the moving axes* XYZ *we only need to add in the equations* (E) *to the real forces acting at a particle* m_i, *two fictitious forces, namely* $-m_i j_{ci}$ *and* $m_i j_{ti}$, *denoting by* m_i *the mass of the particle, by* j_{ci} *its convective acceleration* (i. e., acceleration due to convective motion) *and by* j_{ti} *its turning acceleration.*

I will not stop to demonstrate this principle which follows at once from the well-known theorem of Coriolis.

§ 3. *Differential equations of relative motion.* As a first application of the principle just enunciated, the differential equations of LAGRANGE for relative motion will be derived. Let $\mu = 3n - s$ be the degree of freedom of a given system of n particles subject to s conditions of constraint, and $q_1, q_2, \cdots, q_\mu$ the μ independent coördinate parameters which define the relative position of the system. By reason of the principle of § 2 we can write at once the differential equations of Lagrange for relative motion:

$$(1) \qquad p_k = \frac{\partial T^{(r)}}{\partial q_k'}, \qquad \frac{\partial p_k}{dt} = \frac{\partial T^{(r)}}{\partial q_k} + Q_k, \qquad (k = 1, 2, \cdots, \mu),$$

where $T^{(r)}$ denotes the relative kinetic energy of the system, and

$$(2) \qquad Q_k = \sum_{i=1}^{i=n} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right),$$

provided the forces F_i comprise the fictitious forces of Coriolis as well as the real forces, i. e.,

* See Professor ZIWET'S translation of SOMOF'S *Mechanics*: Kinematics, chap. 17, p. 394.

$$
\begin{aligned}
F_{ix} &= X_i + m_i j_{tix} - m_i j_{cix}, \\
F_{iy} &= Y_i + m_i j_{tiy} - m_i j_{ciy}, \\
F_{iz} &= Z_i + m_i j_{tiz} - m_i j_{ciz},
\end{aligned} \tag{3}
$$

X_i, Y_i, Z_i being the projections of the real force acting on the particle m_i.

§ 4. Let us now transform the expression of Q_k by introducing certain new functions K, L, G, defined as follows:

$$K = -\sum_{i=1}^{i=n} m_i(x_i\dot{x}_{0x} + y_i\dot{v}_{0y} + z_i\dot{x}_{0z}), \tag{4}$$

$$L = \sum_{i=1}^{i=n} m_i[\,p(y_iz_i' - z_iy_i') + q(z_ix_i' - x_iz_i') + r(x_iy_i' - y_ix_i')\,], \tag{5}$$

$$G = \tfrac{1}{2}\sum_{i=1}^{i=n} m_i[\,(qz_i - ry_i)^2 + (rx_i - pz_i)^2 + (py_i - qx_i)^2\,], \tag{6}$$

where $\dot{v}_0$ denotes the acceleration of the origin of the axes of relative coördinates, and, p, q, r, the components of the angular velocity ω of convective rotation.

By means of the familiar formulas:

$$
\begin{aligned}
j_{cx} &= \dot{v}_{0x} + p(px + qy + rz) - \omega^2 x + \left(z\frac{dq}{dt} - y\frac{dr}{dt}\right), \\
j_{cy} &= \dot{v}_{0y} + q(px + qy + rz) - \omega^2 y + \left(x\frac{dr}{dt} - z\frac{dp}{dt}\right), \\
j_{cz} &= \dot{v}_{0z} + r(px + qy + rz) - \omega^2 z + \left(y\frac{dp}{dt} - x\frac{dq}{dt}\right);
\end{aligned}
$$

$$
\begin{aligned}
j_{tx} &= -2(qz' - ry'), \\
j_{ty} &= -2(rz' - px'), \\
j_{tz} &= -2(py' - qx');
\end{aligned}
$$

one can readily verify that

$$\sum_{i=1}^{i=n} m_i\left[(j_{tix} - j_{cix})\frac{\partial x_i}{\partial q_k} + (j_{tiy} - j_{ciy})\frac{\partial y_i}{\partial q_k} + (j_{tiz} - j_{ciz})\frac{\partial z_i}{\partial q_k}\right] = \frac{\partial}{\partial q_k}(K + L + G) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_k'}\right),$$

and consequently,

$$Q_k = Q_k^{(0)} + \frac{\partial}{\partial q_k}(K + L + G) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_k'}\right), \tag{7}$$

where I have put

$$Q_k^{(0)} = \sum_{i=1}^{i=n}\left(X_i\frac{\partial x_i}{\partial q_k} + Y_i\frac{\partial y_i}{\partial q_k} + Z_i\frac{\partial z_i}{\partial q_k}\right). \tag{8}$$

If now we introduce the expression (7) of Q_k into (1) and put

$$\mathfrak{p}_k = p_k + \frac{\partial L}{\partial q_k'} = \frac{\partial (T^{(r)} + L)}{\partial q_k'}, \tag{9}$$

we obtain the required differential equations:

$$\frac{d\mathfrak{p}_k}{dt} = \frac{\partial}{\partial q_k}(T^{(r)} + K + L + G) + Q_k^{(0)} \quad (k=1, 2, \cdots, \mu). \tag{10}$$

§ 5. *Bour's equations.* In particular, if the real forces have a potential function U and if we put for the sake of convenience,

$$U_1 = U + K + G, \tag{11}$$

equations (10) become:

$$\frac{d\mathfrak{p}_k}{dt} = \frac{\partial}{\partial q_k}(T^{(r)} + L + U_1) \qquad (k=1, 2, \cdots, u). \tag{10'}$$

The last equations differ from those obtained by BOUR * only in form. Bour introduces the function T_2:

$$T_2 = \tfrac{1}{2}\sum_{i=1}^{i=n} m_i[(x_i' + qz_i - ry_i)^2 + (y_i' + rx_i - pz_i)^2 + (z_i' + py_i - qx_i)^2]. \tag{12}$$

It is easily seen that

$$T_2 = T^{(r)} + L + G, \tag{13}$$

and since the function G does not contain the variables q_1', q_2', $\cdots$, q_μ',

$$\frac{\partial T_2}{\partial q_k'} = \frac{\partial (T^{(r)} + L)}{\partial q_k'} = \mathfrak{p}_k,$$

so that equations (10′) are identical with the following:

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial q_k'}\right) - \frac{\partial T_2}{\partial q_k} = \frac{\partial (U + K)}{\partial q_k}, \qquad (k=1, 2, \cdots, \mu), \tag{14}$$

given by BOUR.

§ 6. *The functions K, L, G.* These three functions may be thrown into more convenient forms which, moreover, have the advantage of making the results of this chapter applicable to solid bodies without further developments.

From the definitions of K, L and G we have:

$$K = -M\dot{v}_0 r_c \cos(\dot{v}_0, r_c) \tag{4'}$$

* Journal de Liouville, ser. 2, vol. 8, p. 15.

where M denotes the total mass of the material system and r_c the radius vector of its center of inertia;

$$L = \omega \mathfrak{M}_0 \cos(\omega, \mathfrak{M}_0), \tag{5'}$$

where $\mathfrak{M}_0$ denotes the principal moment of relative momentum with regard to the origin of the relative coördinates; and finally, G is the kinetic energy of the material system due to convective rotation, so that for a rigid system or solid body

$$G = \tfrac{1}{2} I_\omega \omega^2, \tag{6'}$$

where I_ω denotes the moment of inertia of the rigid system or solid body with regard to the instantaneous axis (ω).

§7. It will be observed that $K = 0$ in the following cases:

(1) If the origin of the relative coördinates is fixed or moving uniformly in a straight line;

(2) If this origin coincides with the center of inertia of the material system;

(3) If the acceleration of the origin of relative coördinates is at right angles to the radius vector of the center of inertia.

When the convective motion is *translatory*, $\omega = 0$ and consequently $L = G = 0$. The function L vanishes in two other cases:

(1) If the principal moment of relative momentum about the origin of relative coördinates vanishes;

(2) If this moment of relative momentum is at right angles to the instantaneous axis of convective rotation.

CHAPTER II.

Canonical Equations of Hamilton-Jacobi for Relative Motion.

§1. *Differential equations of relative motion.* According to the principle derived from the theorem of Coriolis we can write at once the differential equations of Hamilton-Jacobi for relative motion in the form:

$$(1) \qquad \begin{cases} \dfrac{dp_k}{dt} = -\dfrac{\partial \Theta}{\partial q_k} + Q_k, \qquad \dfrac{dq_k}{dt} = \dfrac{\partial \Theta}{\partial p_k} \qquad (k = 1, 2, \cdots, \mu), \\ \Theta = \sum_{\epsilon=1}^{\epsilon=\mu} p_\epsilon q'_\epsilon - T^{(c)}, \end{cases}$$

where Q_k is again given by formula (7) of the preceding chapter.

§ 2. The variables $\mathfrak{p}_k$ will now be introduced instead of the p_k into formulas (1). Let Θ_1 denote the function Θ after this change of variables is effected. Then

$$\frac{\partial \Theta_1}{\partial q_k} = \frac{\partial \Theta}{\partial q_k} - \sum_{e=1}^{e=\mu} \frac{\partial \Theta}{\partial p_e} \frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial q_e'}\right) = \frac{\partial \Theta}{\partial q_k} - \sum_{e=1}^{e=\mu} q_e' \frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial q_e'}\right)$$

$$= \frac{\partial \Theta}{\partial q_k} - \frac{\partial}{\partial q_k} \sum_{e=1}^{e=\mu} q_e' \frac{\partial L}{\partial q_e'}.$$

On the other hand

$$\sum_{e=1}^{e=\mu} q_e' \frac{\partial L}{\partial q_e'} = L - \lambda, \tag{2}$$

where λ denotes the function

$$\lambda = \sum_{i=1}^{i=n} m_i \left[p\left(y_i \frac{\partial z_i}{\partial t} - z_i \frac{\partial y_i}{\partial t}\right) + q\left(z_i \frac{\partial x_i}{\partial t} - x_i \frac{\partial z_i}{\partial t}\right) + r\left(x_i \frac{\partial y_i}{\partial t} - y_i \frac{\partial x_i}{\partial t}\right)\right]. \tag{3}$$

In the partial derivatives, x_i, y_i, z_i are considered functions of $q_1, q_2, \cdots q_\mu, t$. Hence

$$\frac{\partial \Theta_1}{\partial q_k} = \frac{\partial \Theta}{\partial q_k} - \frac{\partial (L - \lambda)}{\partial q_k},$$

and therefore

$$-\frac{\partial \Theta}{\partial q_k} + Q_k = -\frac{\partial}{\partial q_k}(\Theta_1 - K - G - \lambda) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_k'}\right) + Q_k^{(0)}.$$

Further, it is obvious that

$$\frac{\partial \Theta}{\partial p_k} = \frac{\partial \Theta_1}{\partial \mathfrak{p}_k}.$$

By means of the last two formulas the differential equations (1) may be given the new form:

$$\frac{d\mathfrak{p}_k}{dt} = -\frac{\partial}{\partial q_k}(\Theta_1 - K - G - \lambda) + Q_k^{(0)}, \tag{4}$$

$$\frac{dq_k}{dt} = \frac{\partial \Theta_1}{\partial \mathfrak{p}_k} = \frac{\partial}{\partial \mathfrak{p}_k}(\Theta_1 - K - G - \lambda), \tag{5}$$

This is the required system of differential equations.

§ 3. *Equations of Kamerlingh-Onnes.* In particular, if the real forces have a potential function U we arrive at the formulas given by KAMERLINGH-ONNES:*

* *Over de bettrekkelijke Beweging,* Nieuw Archief voor Wiskunde (Amsterdam), vol. 5.

$$(6)\qquad \begin{cases} \dfrac{dv_k}{dt} = -\dfrac{\partial H}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial H}{\partial v_k}, \\ H = \Theta_1 - U_1 - \lambda, \end{cases}$$

U_1 having the same meaning as in the first chapter.

The mechanical significance of the function λ is similar to that of the function L, the difference being that into λ enter not the whole relative velocities x_i', y_i', z_i', but only those parts of these velocities which would remain if the independent coördinate parameters became constant.

§ 4. *Bour's equations.* If x_i, y_i, z_i are functions of the variables q_1, q_2, $\cdots$, q_μ only, and if the equations of constraint do not involve the time explicitly, we shall have $\lambda = 0$ and $\Theta_1 = T^{(r)}$, so that equations (6) take the form:

$$(7)\qquad \begin{cases} \dfrac{dv_k}{dt} = -\dfrac{\partial H_1}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial H_1}{\partial v_k}, \\ H_1 = T^{(r)} - U_1. \end{cases}$$

These equations were given first by BOUR (loc. cit.).

CHAPTER III.

CONVECTION AS A PERTURBATION. THE PERTURBATIVE FUNCTION OF CONVECTIVE MOTION.

§1. *Convection as a perturbation.* If convection did not exist, the absolute and the relative motions of a given material system would be identical. Convection may therefore be considered as a disturbing element of that motion which the material system would have if the axes XYZ were fixed in space. From this point of view the relative motion of the material system becomes a *disturbed motion*, the *undisturbed motion* being that which would take place if convection did not exist, *i. e.*, if the axis XYZ were fixed in space.

This theory gives rise to a *perturbative function* whose general expression I will now proceed to derive.

§2. *Differential equations of disturbed and undisturbed motion.* It is assumed that the real forces have a potential function U. Then we have to integrate the system:

$$(1)\qquad \begin{cases} \dfrac{dv_k}{dt} = -\dfrac{\partial H}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial H}{\partial v_k} \qquad (k = 1, 2, \cdots, u), \\ H = \Theta_1 - U_1 - \lambda = \Theta_1 - U - K - G - \lambda. \end{cases}$$

If convection did not exist we should have

$$K = L = G = \lambda = 0, \quad \mathfrak{p}_k = p_k, \quad \Theta_1 = \Theta,$$

and the differential equations of undisturbed motion would reduce to

$$\begin{cases} \dfrac{dp_k}{dt} = -\dfrac{\partial E}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial E}{\partial p_k}; \\ E = \Theta - U. \end{cases} \tag{2}$$

§3. *The perturbative function of convective motion.* Let us denote by the symbol [ ] that in the enclosed function the letter p has been replaced by the letter $\mathfrak{p}$.

Now, put

$$H = [E] + [\Omega], \tag{3}$$

and let us see what is the meaning of the function Ω.

First of all it is clear that $\Omega = 0$ when convection does not exist, because in that case $[E] = E = H$.

Further, it is readily seen that the canonical integrals of the differential equations (1) and those of the differential equations:

$$\frac{dp_k}{dt} = -\frac{\partial(E+\Omega)}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial(E+\Omega)}{\partial p_k} \qquad (k = 1, 2, \cdots, \mu), \tag{4}$$

are identical. In fact, the integration of the system (1) is equivalent to solving the partial differential equation:

$$\frac{\partial V}{\partial t} + \{H\}_{\mathfrak{p} = \partial V/\partial q} = 0, \tag{5}$$

indicating by the symbol $\{\ \}_{\mathfrak{p} = \partial V/\partial q}$ that in the expression of H the letter $\mathfrak{p}$ is replaced by the partial derivative $\partial V/\partial q$; and the canonical integrals of the differential equations (1) will be:

$$\frac{\partial V}{\partial a_k} = -\beta_k, \tag{6}$$

where $a_1, a_2, \cdots, a_\mu$ are the arbitrary constants arising from the integration of equation (5) and $\beta_1, \beta_2, \cdots, \beta_\mu$ denote new arbitrary constants. Now to integrate the system (4) we have to solve the partial differential equation:

$$\frac{\partial V}{\partial t} + \{E + \Omega\}_{p = \partial V/\partial q} = 0,$$

which is obviously identical with equation (5) and therefore leads to the same canonical integrals (6).

It follows from the above that Ω is the required perturbative function. It will be called the *perturbative function of convective motion.*

§4. *Solution of the problem when the integrals of undisturbed motion are known.* We have at the same time arrived at the following important result.

Suppose we have integrated the canonical system of differential equations (2) defining the motion of a material system on the assumption that the axes XYZ are fixed in space, and let $\alpha_1, \beta_1; \alpha_2, \beta_2; \cdots; \alpha_\mu, \beta_\mu$ be the canonical system of arbitrary constants. *To obtain the integrals of relative motion of the given material system with regard to the moving axes* XYZ, *we only need to substitute in the integrals of the equations* (2), *instead of the constants* $\alpha_1, \alpha_2, \cdots; \beta_1, \beta_2, \cdots$, *their values as functions of* t *obtained through the integration of the new canonical system of differential equations*:

$$\frac{d\alpha_k}{dt} = \frac{\partial \Omega}{\partial \beta_k}, \quad \frac{d\beta_k}{dt} = -\frac{\partial \Omega}{\partial \alpha_k}, \qquad (k = 1, 2, \cdots, \mu),$$

where Ω is given by equation (3).

It remains to give the general expression of the perturbative function Ω.

§5. *Expression of the perturbative function.* The function Θ introduced in the preceding chapter must be expressed in terms of the variables $q_1, q_2, \cdots, q_\mu; p_1, p_2, \cdots, p_\mu; t$. If therefore we put

$$q_k' = \xi_k + \sum_{e=1}^{e=\mu} h_{ke} p_e, \tag{7}$$

$$T^{(c)} = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} h_{ke} p_k p_e + \tau, \tag{8}$$

where the coefficients h_{ke} and ξ_k, and the function* τ are given functions of $q_1, q_2, \cdots, q_\mu, t$, and $h_{ke} = h_{ek}$, we obtain:

$$\Theta = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} h_{ke} p_k p_e + \sum_{k=1}^{k=\mu} \xi_k p_k - \tau;$$

from which Θ_1 is derived by introducing the variables $\mathfrak{p}_k$ in lieu of the p_k by means of the formula:

$$p_k = \mathfrak{p}_k - \frac{\partial L}{\partial q_k'}.$$

* For the meaning of the function τ see (17) chap. IV and (21) chap. VI.

Thus we find that

$$\Theta_1 = \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\mathfrak{p}_k\mathfrak{p}_e - \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\mathfrak{p}_e\frac{\partial L}{\partial q_k'} + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'} + \sum_{k=1}^{k=\mu}\xi_k\mathfrak{p}_k - \sum_{k=1}^{k=\mu}\xi_k\frac{\partial L}{\partial q_k'} - \tau.$$

Denote for a moment by Θ_1' the function Θ_1 when the letter $\mathfrak{p}$ is replaced by the letter p. Then

$$\Theta_1' = \Theta - \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}p_e\frac{\partial L}{\partial q_k'} - \sum_{k=1}^{k=\mu}\xi_k\frac{\partial L}{\partial q_k'} + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'}$$

$$= \Theta - \sum_{k=1}^{k=\mu} q_k'\frac{\partial L}{\partial q_k'} + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'},$$

or, taking into account (2), chap. II,

$$\Theta_1' = \Theta - L + \lambda + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'}. \tag{9}$$

Now it follows from formula (3) that

$$\Omega = \Theta_1' - U_1 - \lambda - E = \Theta_1' - \Theta - K - G - \lambda,$$

and substituting here the value of $\Theta_1' - \Theta$ from (9) we find:

$$\Omega = -K - L - G + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'}, \tag{10}$$

the desired expression of the perturbative function.

This formula was given first by KAMERLINGH-ONNES (l. c.). Observing that

$$\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_e'} = \frac{\partial L}{\partial p_k},$$

we may give to Ω the slightly different form:

$$\Omega = -K - L - G + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial p_k}. \tag{10'}$$

CHAPTER IV.

DIGRESSION ON SOME TRANSFORMATION FORMULAS IN PASSING FROM CARTESIAN COÖRDINATES TO A SYSTEM OF INDEPENDENT COÖRDINATE PARAMETERS.*

§1. *Preliminaries.* Let us first consider a moving system composed of n particles subject to the s conditions of constraint:

$$\phi_1(u_1, u_2, \cdots, u_{3n}, t) = 0, \quad \phi_2 = 0, \cdots, \quad \phi_s = 0, \tag{1}$$

where

$$u_1 = x_1\sqrt{m_1}, \quad u_4 = x_2\sqrt{m_2}, \quad \cdots \quad u_{3n-2} = x_n\sqrt{m_n},$$
$$u_2 = y_1\sqrt{m_1}, \quad u_5 = y_2\sqrt{m_2}, \quad \cdots \quad u_{3n-1} = y_n\sqrt{m_n},$$
$$u_3 = z_1\sqrt{m_1}, \quad u_6 = z_2\sqrt{m_2}, \quad \cdots \quad u_{3n} = z_n\sqrt{m_n},$$

$m_1, m_2, \cdots, m_n$ denoting, as usual, the masses of the n particles.

Let, further, $q_1, q_2, \cdots, q_\mu$ be a system of independent coördinate parameters, μ being the degree of freedom of the given material system, i. e., $\mu = 3n - s$.

If we put

$$q_i = f_i(u_1, u_2, \cdots, u_{3n}, t) \qquad (i = 1, 2, \cdots, \mu),$$

the functions $f_1, f_2, \cdots, f_\mu$ will involve the time explicitly as well as the functions $\phi_1, \phi_2, \cdots, \phi_s$.

§ 2. *Object of this chapter.* The object of this chapter is to prove the formulas:

$$X_i = \sum_{k=1}^{k=\mu}\sum_{\epsilon=1}^{\epsilon=\mu} A_{k\epsilon}X^{(k)}\frac{\partial u_i}{\partial q_\epsilon} + \sum_{k=1}^{k=s}\sum_{\epsilon=1}^{\epsilon=s}\Phi_{k\epsilon}X(\phi_k)\frac{\partial\phi_\epsilon}{\partial u_i}, \tag{2}$$

$$\sum_{i=1}^{i=3n}(X_i)^2 = \sum_{k=1}^{k=\mu}\sum_{\epsilon=1}^{\epsilon=\mu}A_{k\epsilon}X^{(k)}X^{(\epsilon)} + \sum_{k=1}^{k=s}\sum_{\epsilon=1}^{\epsilon=s}\Phi_{k\epsilon}X(\phi_k)X(\phi_\epsilon), \tag{3}$$

where the several symbols have the following meanings: $X_1, X_2, \cdots, X_{3n}$ are certain functions which will be determined later; $X^{(k)}$ and $X(\phi_k)$ are defined by the formulas:

$$X^{(k)} = \sum_{i=1}^{i=3n} X_i\frac{\partial u_i}{\partial q_k}, \tag{a}$$

$$X(\phi_k) = \sum_{i=1}^{i=3n} X_i\frac{\partial\phi_k}{\partial u_i}; \tag{b}$$

* See the note at the close of this chapter.

while to define A_{ke} and Φ_{ke} we have the formulas:

$$(c)\qquad \begin{cases} A_{ke} = A_{ek} = \dfrac{\partial \log A}{\partial a_{ke}} = \dfrac{\partial \log A}{\partial a_{ek}}, \\ A = |\, a_{ek} \,| \qquad (e, k = 1, 2, \cdots, \mu), \\ a_{ek} = a_{ke} = \displaystyle\sum_{i=1}^{i=3n} \frac{\partial u_i}{\partial q_k} \frac{\partial u_i}{\partial q_e}, \end{cases}$$

$$(d)\qquad \begin{cases} \Phi_{ke} = \Phi_{ek} = \dfrac{\partial \log \Phi}{\partial \phi_{ke}} = \dfrac{\partial \log \Phi}{\partial \phi_{ek}}, \\ \Phi = |\, \phi_{ek} \,| \qquad (e, k = 1, 2, \cdots, \mu), \\ \phi_{ek} = \phi_{ke} = \displaystyle\sum_{i=1}^{i=3n} \frac{\partial \phi_k}{\partial u_i} \frac{\partial \phi_e}{\partial u_i}. \end{cases}$$

§ 3. *Proof of the proposed formulas* (2), (3). Let δu_i, δq_k denote the variations of u_i and of q_k respectively. The $\delta q_1, \delta q_2, \cdots, \delta q_\mu$ will be arbitrary, while the $\delta u_1, \delta u_2, \cdots, \delta u_{3n}$ must satisfy the s conditions:

$$(4)\qquad 0 = \sum_{i=1}^{i=3n} \frac{\partial \phi_e}{\partial u_i} \delta u_i \qquad (e = 1, 2, \cdots, s).$$

I shall now form the expression of the function

$$\sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \delta q_k$$

in terms of the variations $\delta u_1, \delta u_2, \cdots, \delta u_{3n}$. Two such expressions will be obtained, and by comparing them the formulas (2) will be proved.

To begin with, it is obvious that

$$(5)\qquad \sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \delta q_k = \delta u_i,$$

which is the first of the two desired expressions. On the other hand, the variations $\delta q_1, \delta q_2, \cdots, \delta q_\mu$ may be expressed in terms of the variations $\delta u_1, \delta u_2, \cdots, \delta u_{3n}$ as follows:

$$\delta q_k = \sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i} \delta u_i;$$

and thus the second of the two desired expressions is now obtained, namely,

$$(6)\qquad \sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \delta q_k = \sum_{k=1}^{k=\mu} \sum_{j=1}^{j=3n} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_j} \delta u_j.$$

The right-hand sides of (5) and of (6) must be equal for all variations δu_1, δu_2, $\cdots$, δu_{3n} which satisfy the s conditions (4). Hence, introducing certain factors ρ_{ei} which will be determined later, we may write down the following equations:

$$\delta u_i = \sum_{k=1}^{k=\mu} \sum_{j=1}^{j=3n} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_j} \delta u_j + \sum_{e=1}^{e=s} \sum_{j=1}^{j=3n} \rho_{ei} \frac{\partial \phi_e}{\partial u_j} \delta u_j \qquad (i=1, 2, \cdots, 3n), \tag{7}$$

from which we obtain the relations:*

$$\sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_i} = 1 - \sum_{e=1}^{e=s} \rho_{ei} \frac{\partial \phi_e}{\partial u_i}, \tag{8}$$

$$\sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_j} = - \sum_{e=1}^{e=s} \rho_{ei} \frac{\partial \phi_e}{\partial u_j} \qquad (j \neq i). \tag{9}$$

With the notations of §2, and with the following:

$$c_{ke} = \sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i} \frac{\partial \phi_e}{\partial u_i}, \tag{e}$$

the factors ρ_{ei} in equations (8), (9) are readily obtained in the form †

$$\rho_{ei} = \sum_{m=1}^{m=s} \Phi_{me} \left(\frac{\partial \phi_m}{\partial u_i} - \sum_{k=1}^{k=\mu} c_{km} \frac{\partial u_i}{\partial q_k} \right).$$

Introducing these expressions into formulas (8) and (9) we find:

$$\sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_i} = 1 - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Phi_{me} \frac{\partial \phi_m}{\partial u_i} \frac{\partial \phi_e}{\partial u_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Phi_{me} c_{km} \frac{\partial u_i}{\partial q_k} \frac{\partial \phi_e}{\partial u_i}, \tag{10}$$

$$\sum_{k=1}^{k=\mu} \frac{\partial u_j}{\partial q_k} \frac{\partial q_k}{\partial u_i} = - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Phi_{me} \frac{\partial \phi_m}{\partial u_j} \frac{\partial \phi_e}{\partial u_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Phi_{me} c_{km} \frac{\partial u_j}{\partial q_k} \frac{\partial \phi_e}{\partial u_i} \qquad (j \neq i). \tag{11}$$

---

* Compare with the relations:

$$\sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i} \frac{\partial u_i}{\partial q_k} = 1, \qquad \sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i} \frac{\partial u_i}{\partial q_e} = 0 \qquad (k \neq e),$$

which follow from the definition of $q_1, q_2, \cdots, q_\mu$.

† Proceed as follows: multiply both sides of (8) and of (9) by $\partial \phi_m / \partial u_i$ and by $\partial \phi_m / \partial u_j$ respectively; then add together the resulting equations, thus obtaining the s equations:

$$\sum_{k=1}^{k=\mu} \sum_{j=1}^{j=3n} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_j} \frac{\partial \phi_m}{\partial u_j} = \sum_{k=1}^{k=\mu} c_{km} \frac{\partial u_i}{\partial q_k} = \frac{\partial \phi_m}{\partial u_i} - \sum_{e=1}^{e=s} \rho_{ei} \phi_{em} \qquad (m=1, 2, \cdots, s),$$

from which the above expression of ρ_{ei} is readily derived.

§ 4. Now, let $X_1, X_2, \cdots, X_{3n}$ be certain functions to be fixed later. Multiply both sides of (10) and of (11) by X_i and by X_j respectively, then give j all the values $1, 2, \cdots, 3n$ except i and add together the resulting equations. The relations so obtained will be:

$$(12)\quad \sum_{k=1}^{k=\mu} X^{(k)} \frac{\partial q_k}{\partial u_i} = X_i - \sum_{m=1}^{m=s}\sum_{e=1}^{e=s} \Phi_{me} X(\phi_m) \frac{\partial \phi_e}{\partial u_i} + \sum_{m=1}^{m=s}\sum_{e=1}^{e=s}\sum_{k=1}^{k=\mu} \Phi_{me} c_{km} X^{(k)} \frac{\partial \phi_e}{\partial u_i} \qquad (i = 1, 2, \cdots, 3n).$$

§ 4 bis. A first selection of the functions $X_1, X_2, \cdots, X_{3n}$ will now be made. Let $X_i = \partial u_i / \partial q_r$. Then

$$X^{(k)} = \sum_{i=1}^{i=3n} \frac{\partial u_i}{\partial q_r}\frac{\partial u_i}{\partial q_k} = a_{kr},$$

$$X(\phi_m) = \sum_{i=1}^{i=3n} \frac{\partial \phi_m}{\partial u_i}\frac{\partial u_i}{\partial q_r} = 0 \qquad (m = 1, 2, \cdots, s);$$

and, therefore, by (12),

$$(13)\quad \sum_{k=1}^{k=\mu} a_{kr} \frac{\partial q_k}{\partial u_i} = \frac{\partial u_i}{\partial q_r} + \sum_{k=1}^{k=\mu}\sum_{m=1}^{m=s}\sum_{e=1}^{e=s} \Phi_{me} c_{km} a_{kr} \frac{\partial \phi_e}{\partial u_i} \qquad (r = 1, 2, \cdots, \mu).$$

From the μ equations (13) we obtain

$$(14)\quad \frac{\partial q_k}{\partial u_i} = \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial u_i}{\partial q_e} + \sum_{m=1}^{m=s}\sum_{e=1}^{e=s} \Phi_{me} c_{km} \frac{\partial \phi_e}{\partial u_i} \qquad (k = 1, 2, \cdots, \mu).$$

Equations (13), (14) may be found in a paper by D. BOBYLEF.*

§ 5. Substituting the expressions (14) of $\partial q_k / \partial u_i$ into formula (12) the latter becomes:

$$(2)\quad X_i = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} X^{(k)} \frac{\partial u_i}{\partial q_e} + \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} X(\phi_k) \frac{\partial \phi_e}{\partial u_i} \qquad (i = 1, 2, \cdots, 3n),$$

which is the first of the two formulas we proposed to prove.

Let now $Y_1, Y_2, \cdots, Y_{3n}$ be a new set of functions left for the present indeterminate. If we multiply both sides of (2) by Y_i and add all the resulting equations ($i = 1, 2, \cdots, 3n$) we shall have the formula:

$$(15)\quad \sum_{i=1}^{i=3n} X_i Y_i = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} X^{(k)} Y^{(e)} + \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \phi_{ke} X(\phi_k) Y(\phi_e),$$

where

$$(f')\quad Y^{(k)} = \sum_{i=1}^{i=3n} Y_i \frac{\partial u_i}{\partial q_k}, \quad Y(\phi_k) = \sum_{i=1}^{i=3n} Y_i \frac{\partial \phi_k}{\partial u_i}.$$

* *On the change of coördinates in the differential equations of dynamics.* Supplement to the 58th vol. of the Memoirs (Zapiski) of the Imperial Academy of Sciences of St. Petersburg; 1888 (Russian).

In particular, if $X_i = Y_i$ $(i = 1, 2, \cdots, 3n)$, (15) reduces to (3) which is thus proved.

§6. *Kinetic energy of a material system. Bobylef's formula.* To conclude, I will make an application of formulas (2) and (3).

Let $A_i = du_i/dt$. Then we shall have:

$$X(\phi_k) = \sum_{i=1}^{i=3n} \frac{\partial \phi_k}{\partial u_i} \frac{du_i}{dt} = -\frac{\partial \phi_k}{\partial t},$$

$\partial \phi_k/\partial t$ denoting the partial derivative of the function ϕ_k with respect to the time; and, further,

$$X^{(k)} = \sum_{i=1}^{i=3n} \frac{\partial u_i}{\partial q_k} \frac{du_i}{dt} = \sum_{i=1}^{i=3n} \frac{\partial u_i'}{\partial q_k'} u_i' = \frac{\partial T}{\partial q_k'} = p_k,$$

where $T = \frac{1}{2} \sum_{i=1}^{i=3n} u_i'^2$. Hence, by formula (2),

$$\frac{du_i}{dt} = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial u_i}{\partial q_k} p_e - \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_k}{\partial t} \frac{\partial \phi_e}{\partial u_i}, \tag{16}$$

and, by formula (3),

$$T = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} p_k p_e + \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_k}{\partial t} \frac{\partial \phi_e}{\partial t}. \tag{17}$$

Formula (17) was given for the first time by D. BOBYLEF (loc. cit.).

Note. In deriving the formulas of this chapter no mention was made as to whether the axes XYZ were fixed in space or moving. Indeed, the results here obtained are entirely independent of such considerations: x, y, z may be either *absolute* or *relative* coördinates. A similar remark must be made with regard to the result of chapter VI.

CHAPTER V.

APPLICATION OF THE FORMULAS OF THE PRECEDING CHAPTER TO THE PERTURBATIVE FUNCTION OF CONVECTIVE MOTION.

§1. *The function G_2.* If we put

$$G_2 = G - \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} h_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'}, \tag{1}$$

the perturbative function of convective motion takes the form:

$$\Omega = -K - L - G_2. \tag{2}$$

Systems consisting of n particles will be first considered and the function G_2 for such systems will be transformed by means of formula (3) of the preceding chapter.

To begin with I remark that for a system of n particles the coefficients h_{ke} are identical with the coefficients denoted by A_{ke} in the preceding chapter, so that

$$G_2 = G - \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'}. \tag{1'}$$

Let us now put $X_i = \partial L/\partial u_i'$ in formula (3), chapter IV. Then we shall have:

$$X^{(k)} = \sum_{i=1}^{i=3n}\frac{\partial L}{\partial u_i'}\frac{\partial u_i}{\partial q_k} = \sum_{i=1}^{i=3n}\frac{\partial L}{\partial u_i'}\frac{\partial u_i'}{\partial q_k'} = \frac{\partial L}{\partial q_k'},$$

$$\sum_{i=1}^{i=3n}(X_i)^2 = \sum_{i=1}^{i=3n}\left(\frac{\partial L}{\partial u_i'}\right)^2 = 2G;$$

and therefore, by formula (3),

$$G = \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'} + \tfrac{1}{2}\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Phi_{ke}L(\phi_k)L(\phi_e), \tag{3}$$

where I have put

$$L(\phi_k) = \sum_{i=1}^{i=3n}\frac{\partial L}{\partial u_i'}\frac{\partial \phi_k}{\partial u_i}. \tag{4}$$

Substituting the expression (3) of G into formula (1') we find that

$$G_2 = {}^1\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Phi_{ke}L(\phi_k)L(\phi_e). \tag{5}$$

§2. *Perturbative function for a free system of particles.* It follows from this expression of the function G_2 that $G_2 = 0$ when $s = 0$. Hence this theorem due to KAMERLINGH-ONNES but not rigorously proved by him:

For a free system of particles the perturbative function of convective motion reduces to $-(K+L)$.

§ 3. *Perturbative function for a freely moving rigid system.* Another interesting case when $G_2 = 0$ is that of a freely moving rigid system, as will presently be shown. First it may be remarked that

$$\begin{aligned}\sum_{i=1}^{i=3n}\frac{\partial L}{\partial u_i'}\frac{\partial \phi_k}{\partial u_i} &= \sum_{i=1}^{i=3n}\frac{1}{m_i}\left(\frac{\partial L}{\partial x_i'}\frac{\partial \phi_k}{\partial x_i} + \frac{\partial L}{\partial y_i'}\frac{\partial \phi_k}{\partial y_i} + \frac{\partial L}{\partial z_i'}\frac{\partial \phi_k}{\partial z_i}\right)\\ &= \sum_{i=1}^{i=3n}\left((qz_i - ry_i)\frac{\partial \phi_k}{\partial x_i} + (rx_i - pz_i)\frac{\partial \phi_k}{\partial y_i} + (py_i - qx_i)\frac{\partial \phi_k}{\partial z_i}\right)\\ &= p\sum_{i=1}^{i=3n}\left(y_i\frac{\partial \phi_k}{\partial z_i} - z_i\frac{\partial \phi_k}{\partial y_i}\right) + q\sum_{i=1}^{i=3n}\left(z_i\frac{\partial \phi_k}{\partial x_i} - x_i\frac{\partial \phi_k}{\partial z_i}\right)\\ &\qquad + r\sum_{i=1}^{i=3n}\left(x_i\frac{\partial \phi_k}{\partial y_i} - y_i\frac{\partial \phi_k}{\partial x_i}\right),\end{aligned}$$

and therefore, if we put for the sake of convenience,

$$(6)\qquad \begin{cases} R_x^{(k)} = \sum_{i=1}^{i=3n} \left(y_i \dfrac{\partial \phi_k}{\partial z_i} - z_i \dfrac{\partial \phi_k}{\partial y_i} \right), \\ R_y^{(k)} = \sum_{i=1}^{i=3n} \left(z_i \dfrac{\partial \phi_k}{\partial x_i} - x_i \dfrac{\partial \phi_k}{\partial z_i} \right), \\ R_z^{(k)} = \sum_{i=1}^{i=3n} \left(x_i \dfrac{\partial \phi_k}{\partial y_i} - y_i \dfrac{\partial \phi_k}{\partial x_i} \right), \end{cases}$$

we shall have:

$$(7)\qquad L(\phi_k) = pR_x^{(k)} + qR_y^{(k)} + rR_z^{(k)};$$

or, denoting by $R^{(k)}$ a vector whose projections on the axes XYZ are $R_x^{(k)}$, $R_y^{(k)}$, $R_z^{(k)}$ respectively,

$$(8)\qquad L(\phi_k) = \omega R^{(k)} \cos(\omega, R^{(k)}).$$

This expression of $L(\phi_k)$ shows that $L(\phi_k)$ vanishes with $R^{(k)}$, i. e., when

$$(9)\qquad R_x^{(k)} = R_y^{(k)} = R_z^{(k)} = 0.$$

Conditions (9) will be satisfied if the constraint ϕ_k is given by an equation of the form:

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} = \text{constant}.$$

If *all* the constraints are of this form, then we have a freely moving rigid system of n particles ($\mu = 6$, $s = 3n - 6$); and since in this case $L(\phi_1) = L(\phi_2) = \cdots = L(\phi_s) = 0$, the function G_2 will vanish. Thus I have proved the following theorem:

For a freely moving rigid system of particles the perturbative function of convective motion reduces to $-(K + L)$.

§ 4. *The perturbative function for very small values of* ω. A glance at formulas (4), (5) of chap. I and (2), (5), (7) of this chapter shows that

The perturbative function Ω *is the sum of three functions* $-K$, $-L$, $-G_2$, *all three homogeneous in* p, q, r. *The degree of homogeneity is* 0 *for* K, 1 *for* L, 2 *for* G_2.

Sometimes it is convenient to take ω as a small quantity of the first order (as in the case of motion on the surface of the earth). The function L will then be of the first order in ω, the function G_2 of the second. Looking upon Ω as a function of ω we can say that

The perturbative function of the second order in ω *is due solely to the existence of constraints. It vanishes when the system is free or when the particles form a freely moving rigid system.*

§ 5. *Cases when* $G_2 = 0$. More generally, if we observe that *the vector* $R^{(k)}$ *is parallel and proportional to the principal moment of the reactions due to the constraint* ϕ_k, it will be clear that $L(\phi_k) = 0$ when this moment vanishes or when its direction forms a right angle with the direction of the instantaneous axis (ω). Hence,

$G_2 = 0$ *when the principal moments of the reactions due to the constraints* $\phi_1, \phi_2, \cdots, \phi_s$, *either vanish or are at right angles to the direction of the instantaneous axis* (ω).

This theorem includes the case considered in § 3.

§ 6. *Case of a solid body. Expression of the function* G. By a process commonly used in mechanics we at once pass from rigid systems to solid bodies.

Let $\eta_1, \eta_2, \cdots, \eta_6$ be any system of six coördinate parameters which define the position of a free solid body with regard to the invariable * system XYZ and such that the coördinates x, y, z of any point of the body be functions of $\eta_1, \eta_2, \cdots, \eta_6$ only, i. e., these functions *do not involve the time explicitly.* Then, as is well known,

$$T = \tfrac{1}{2} \int (x'^2 + y'^2 + z'^2) dm \tag{10}$$

will be a homogeneous function of the second degree in $\eta_1', \eta_2', \cdots, \eta_6'$, and therefore,

$$2T = \sum_{i=1}^{i=6} \frac{\partial T}{\partial \eta_i'} \eta_i'. \tag{11}$$

Now, let us put

$$\mu_{ij} = \mu_{ji} = \int \left(\frac{\partial x}{\partial \eta_i} \frac{\partial x}{\partial \eta_j} + \frac{\partial y}{\partial \eta_i} \frac{\partial y}{\partial \eta_j} + \frac{\partial z}{\partial \eta_i} \frac{\partial z}{\partial \eta_j} \right) dm. \tag{12}$$

Then, as is readily seen from formulas (10) and (12),

$$\frac{\partial T}{\partial \eta_i'} = \int \left(x' \frac{\partial x}{\partial \eta_i} + y' \frac{\partial y}{\partial \eta_i} + z' \frac{\partial z}{\partial \eta_i} \right) dm = \sum_{j=1}^{j=6} \mu_{ij} \eta_j';$$

and, therefore, by formula (11),

$$T = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \eta_i' \eta_j'. \tag{13}$$

The coefficients μ_{ij} given by formula (12) take the place, in the case of solid bodies, of the coefficients a_{ek} introduced in chapter IV for systems of particles. Likewise to the coefficients A_{ke} will now correspond certain coefficients M_{ij} defined as follows:

* Not to be confused with the term *fixed.* An invariable system is *not deformable* but it may be moving.

$$(14)\qquad \begin{cases} M_{ij} = M_{ji} = \dfrac{\partial \log M}{\partial \mu_{ij}} = \dfrac{\partial \log M}{\partial \mu_{ji}}, \\ M = |\mu_{ij}| \qquad (i, j = 1, 2, \cdots 6). \end{cases}$$

§7. We have seen that for a freely moving rigid system of n particles $G_2 = 0$, and therefore, by (3), for such a system,

$$(15)\qquad G = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'},$$

where now $\mu = 6$. Passing in the familiar way from rigid systems to solid bodies we only have to replace in formula (15) the coefficients A_{ke} by the coefficients M_{ke} and the letter q by the letter η; so that for a free solid body,

$$(16)\qquad G = \tfrac{1}{2} \sum_{k=1}^{k=6} \sum_{e=1}^{e=6} M_{ke} \frac{\partial L}{\partial \eta_k'} \frac{\partial L}{\partial \eta_e'}.$$

Thus we have the following theorem:

If $\eta_1, \eta_2, \cdots, \eta_6$ be any system of coördinate parameters which define the position of a free solid body with regard to the invariable system XYZ; and if the kinetic energy of the body be given by an expression of the form (13); then the function G will be given by formula (16) where the coefficients M_{ke} are derived from the coefficients μ_{ke}, functions of $\eta_1, \eta_2, \cdots, \eta_6$, by means of the relations (14).

§8. *Perturbative function for a free solid body.* We may now extend the results of §3 to solid bodies, namely:

For a free solid body the perturbative function of convective motion reduces to $-(K + L)$.

CHAPTER VI.

Another digression on Transformations of Coördinates in the Equations of Dynamics.*

§1. *Preliminaries.*—We will now pass on to the motion of a solid body subject to s conditions of constraint:

$$(1)\qquad \psi_1(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, t) = 0, \ \psi_2 = 0, \ \cdots, \ \psi_s = 0,$$

where $\eta_1, \cdots, \eta_6$ form a system of coördinates defined as in the preceding chapter. Let again,

*See the note at the end of chapter IV.

$$T = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \eta_i' \eta_j' \tag{2}$$

be the kinetic energy of the body, the coefficients μ_{ij} being given functions of $\eta_1, \cdots, \eta_6$. Let, further, $q_1, \cdots, q_\mu$ be a system of independent coördinate parameters, μ being the degree of freedom of the body, i. e., $\mu = 6 - s$. If we put

$$q_1 = f_1(\eta_1, \cdots, \eta_6, t),$$
$$\cdots\cdots\cdots\cdots$$
$$q_\mu = f_\mu(\eta_1, \cdots, \eta_6, t),$$

the functions $f_1, f_2, \cdots, f_\mu$, as well as the functions $\psi_1, \cdots, \psi_s$, will involve the time explicitly.

§2. *Object of this chapter. Proof of the proposed formulas.*—The object of this chapter is to prove the three formulas (13), (14) and (15) given below. To this end let $\delta\eta_i$, δq_k denote respectively the variations of η_i and of q_k $(i = 1, 2, \cdots, 6; k = 1, 2, \cdots, \mu)$. The variations $\delta q_1, \cdots, \delta q_\mu$ will be arbitrary while the variations $\delta\eta_1, \cdots, \delta\eta_6$ must satisfy the s conditions:

$$0 = \sum_{i=1}^{i=6} \frac{\partial \psi_e}{\partial \eta_i} \delta\eta_i \qquad (e = 1, 2, \cdots, s). \tag{3}$$

If we consider the expression:

$$\sum_{k=1}^{k=\mu} \frac{\partial \eta_i}{\partial q_k} \delta q_k,$$

after the variations $\delta\eta_1, \delta\eta_2, \cdots, \delta\eta_6$ are introduced instead of the variations $\delta q_1, \cdots, \delta q_\mu$, then, by a process similar to that employed in chapter IV, we arrive at the formulas:

$$\sum_{k=1}^{k=\mu} \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_i} = 1 - \sum_{e=1}^{e=s} \rho_{ei} \frac{\partial \psi_e}{\partial \eta_i}, \tag{4}$$

$$\sum_{k=1}^{k=\mu} \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_j} = \quad - \sum_{e=1}^{e=s} \rho_{ei} \frac{\partial \psi_e}{\partial \eta_j}, \qquad (j \neq i), \tag{5}$$

in which the ρ_{ei} denote coefficients to be determined presently.

§3. I shall use, besides the notations (14) of the preceding chapter, also the following:

$$(a) \quad \begin{cases} b_{ke} = b_{ek} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \frac{\partial \eta_i}{\partial q_k} \frac{\partial \eta_j}{\partial q_e}, \\ B = |b_{ke}| \qquad (k, e = 1, 2, \cdots, \mu), \\ B_{ke} = B_{ek} = \frac{\partial \log B}{\partial b_{ke}} = \frac{\partial \log B}{\partial b_{ek}}; \end{cases}$$

$$(b) \qquad d_{me} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \frac{\partial \psi_m}{\partial \eta_i} \frac{\partial q_e}{\partial \eta_j};$$

$$(c) \qquad \begin{cases} \psi_{ke} = \psi_{ek} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \frac{\partial \psi_k}{\partial \eta_i} \frac{\partial \psi_e}{\partial \eta_j}, \\ \Psi = |\psi_{ke}| \qquad (k, e = 1, 2, \cdots, s), \\ \Psi_{ke} = \Psi_{ek} = \frac{\partial \log \Psi}{\partial \psi_{ke}} = \frac{\partial \log \Psi}{\partial \psi_{ek}}. \end{cases}$$

With these notations we readily find the following expressions for the coefficients ρ_{ei}: *

$$\rho_{ei} = \sum_{m=1}^{m=s} \Psi_{me} \left[\sum_{n=1}^{n=6} M_{in} \frac{\partial \psi_m}{\partial \eta_n} - \sum_{k=1}^{k=\mu} d_{mk} \frac{\partial \eta_i}{\partial q_k} \right];$$

substituting these expressions, the relations (4) and (5) become:

$$(6) \qquad \sum_{k=1}^{k=\mu} \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_i} = 1 - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{in} \frac{\partial \psi_m}{\partial \eta_n} \frac{\partial \psi_e}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Psi_{me} d_{mk} \frac{\partial \eta_i}{\partial q_k} \frac{\partial \psi_e}{\partial \eta_i},$$

$$(7) \qquad \sum_{k=1}^{k=\mu} \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_j} = - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{jn} \frac{\partial \psi_m}{\partial \eta_n} \frac{\partial \psi_e}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Psi_{me} d_{mk} \frac{\partial \eta_j}{\partial q_k} \frac{\partial \psi_e}{\partial \eta_i} \qquad (j \neq i).$$

§ 4. Let again $X_1, X_2, \cdots, X_6$ denote certain functions to be fixed later. For the sake of convenience put

$$(d) \qquad X^{(k)} = \sum_{i=1}^{i=6} X_i \frac{\partial \eta_i}{\partial q_k}, \qquad \mathfrak{X}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \cdot X_j \frac{\partial \eta_i}{\partial q_k};$$

$$(e) \qquad X(\psi_k) = \sum_{i=1}^{i=6} X_i \frac{\partial \psi_k}{\partial \eta_i}, \qquad \mathfrak{X}(\psi_k) = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \cdot X_j \frac{\partial \psi_k}{\partial \eta_i}.$$

If we multiply both sides of (6) by X_i and both sides of (7) by X_j, then give j all the values $1, \cdots, 6$, except i, and add together all the resulting equations, we shall obtain:

$$(8) \qquad \begin{cases} X_i = \sum_{k=1}^{k=\mu} X^{(k)} \frac{\partial q_k}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \psi_{me} \mathfrak{X}(\psi_m) \frac{\partial \psi_e}{\partial \eta_i} - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \psi_{me} d_{mk} X^{(k)} \frac{\partial \psi_e}{\partial \eta_i} \\ (i = 1, 2, \cdots, 6). \end{cases}$$

* Proceed as follows:—multiply both sides of equations (4) and (5) by $M_{in}\, \partial \psi_m / \partial \eta_n$ and $M_{jn}\, \partial \psi_m / \partial \eta_n$ respectively; let j take all the values $1, 2, \cdots, 6$ except i; add together all the resulting equations and sum with regard to the index n; then the equations:

$$\sum_{k=1}^{k=\mu} d_{mk} \frac{\partial \eta_i}{\partial q_k} = \sum_{n=1}^{n=6} M_{in} \frac{\partial \psi_m}{\partial \eta_n} - \sum_{e=1}^{e=s} \rho_{ei} \psi_{me} \qquad (i = 1, 2, \cdots, 6):$$

will be obtained, from which at once follow the above expressions for the coefficients ρ_{ei}.

Likewise, if after interchanging the letters i and j in (7) we multiply both sides of (6) by X_i and both sides of (7) by X_j, give j all the values $1, 2, \cdots, 6$ except i, and add together all the resulting equations; then we obtain:

$$(9)\qquad \begin{cases} X_i = \sum_{j=1}^{j=6} \sum_{k=1}^{k=\mu} X_j \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_j} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \psi_{me} M_{in} X(\psi_e) \frac{\partial \psi_m}{\partial \eta_n} \\ \qquad - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \psi_{me} d_{mk} X(\psi_e) \frac{\partial \eta_i}{\partial q_k} \qquad (i = 1, 2, \cdots, 6). \end{cases}$$

Finally, substitute the function $\sum_{j=1}^{j=6} X_j \mu_{ij}$ for the function X_i in (8); then $X^{(k)}$ will change into $\mathfrak{X}^{(k)}$, $\mathfrak{X}(\psi_m)$ into $X(\psi_m)$, and formula (8) will become:

$$(10)\qquad \begin{cases} \sum_{j=1}^{j=6} \mu_{ij} X_j = \sum_{k=1}^{k=\mu} \mathfrak{X}^{(k)} \frac{\partial q_k}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} X(\psi_m) \frac{\partial \psi_e}{\partial \eta_i} \\ \qquad - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Psi_{me} d_{mk} \mathfrak{X}^{(k)} \frac{\partial \psi_e}{\partial \eta_i} \qquad (i = 1, 2, \cdots, 6). \end{cases}$$

§ 4 bis. I will now make a first selection of the functions $X_1, X_2, \cdots, X_6$. Let $X_i = \partial \eta_i / \partial q_n$. Then

$$X(\psi_m) = \sum_{j=1}^{j=6} \frac{\partial \psi_m}{\partial \eta_j} \frac{\partial \eta_j}{\partial q_n} = 0 \qquad (m = 1, 2, \cdots, s),$$

$$\mathfrak{X}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \frac{\partial \eta_i}{\partial q_k} \frac{\partial \eta_j}{\partial q_n} = b_{kn};$$

and therefore, by formula (10),

$$(11)\qquad \sum_{j=1}^{j=6} \mu_{ij} \frac{\partial \eta_j}{\partial q_n} = \sum_{k=1}^{k=\mu} b_{kn} \frac{\partial q_k}{\partial \eta_i} - \sum_{k=1}^{k=\mu} \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} d_{mk} b_{kn} \frac{\partial \psi_e}{\partial \eta_i}.$$

Solving the μ equations obtained by making $n = 1, 2, \cdots, \mu$ in formula (11), with regard to the $\partial q_1 / \partial \eta_i, \cdots, \partial q_\mu / \partial \eta_i$, we find that

$$(12)\qquad \frac{\partial q_k}{\partial \eta_i} = \sum_{j=1}^{j=6} \sum_{e=1}^{e=\mu} \mu_{ij} B_{ke} \frac{\partial \eta_j}{\partial q_e} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} d_{mk} \frac{\partial \psi_e}{\partial \eta_i} \quad \begin{pmatrix} k = 1, 2, \cdots, \mu \\ i = 1, 2, \cdots, 6 \end{pmatrix}.$$

§5. If we substitute the expressions (12) of the $\partial q_k / \partial \eta_i$ in formula (9) the latter becomes:

$$(13)\quad X_i = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{X}^{(e)} \frac{\partial \eta_i}{\partial q_k} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{in} X(\psi_e) \frac{\partial \psi_m}{\partial \eta_n} \quad (i = 1, 2, \cdots, 6).$$

The same substitution made in formula (10) yields the following result:

$$(14)\qquad \begin{cases} \sum_{j=1}^{j=6} X_j \mu_{ij} = \sum_{j=1}^{j=6} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{X}^{(k)} \mu_{ij} \dfrac{\partial \eta_j}{\partial q_e} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} X(\psi_m) \dfrac{\partial \psi_e}{\partial \eta_i} \\ (i = 1, 2, \cdots, 6). \end{cases}$$

Finally, multiplying both sides of (14) by X_i and summing the result with regard to the index i, we obtain a third formula:

$$(15)\qquad \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} X_i X_j = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{X}^{(k)} \mathfrak{X}^{(e)} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} X(\psi_m) X(\psi_e).$$

Formulas (13), (14) and (15) are the three which we proposed to derive.

§6. *Kinetic energy of a solid body.* As an application of these formulas let us put $X_i = d\eta_i/dt$. Then

$$X(\psi_m) = \sum_{i=1}^{i=6} \frac{\partial \psi_m}{\partial \eta_i} \frac{d\eta_i}{dt} = -\frac{\partial \psi_m}{\partial t},$$

where $\partial \psi_m/\partial t$ is the partial derivative of the function ψ_m with regard to the time; further,

$$\mathfrak{X}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \eta_i' \frac{\partial \eta_j}{\partial q_k} = \frac{\partial T}{\partial q_k'} = p_k;$$

and, therefore, by (13), (14), (15),

$$(16)\qquad \eta_i' = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \frac{\partial \eta_i}{\partial q_k} p_e - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{in} \frac{\partial \psi_e}{\partial t} \frac{\partial \psi_m}{\partial \eta_n},$$

$$(17)\qquad \frac{\partial T}{\partial \eta_i'} = \sum_{j=1}^{j=6} \mu_{ij} \eta_j' = \sum_{e=1}^{e=\mu} \sum_{k=1}^{k=\mu} \sum_{j=1}^{j=6} B_{ke} \mu_{ij} \frac{\partial \eta_j}{\partial q_e} p_k - \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \frac{\partial \psi_k}{\partial t} \frac{\partial \psi_e}{\partial \eta_i},$$

$$(18)\qquad T = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \eta_i' \eta_j' = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} p_k p_e + \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \frac{\partial \psi_k}{\partial t} \frac{\partial \psi_e}{\partial t}.$$

§7. *A modification of the three proposed formulas.* If we substitute the function

$$\sum_{j=1}^{j=6} M_{ij} X_j$$

for X_i in formulas (13)–(15) and observe that this will change $\mathfrak{X}^{(k)}$ into $X^{(k)}$, $X(\psi_k)$ into $\mathfrak{X}(\psi_k)$, we shall obtain:

$$(19)\qquad \begin{cases} \sum_{j=1}^{j=6} M_{ij} X_j = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} X^{(e)} \dfrac{\partial \eta_j}{\partial q_k} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{in} \mathfrak{X}(\psi_e) \dfrac{\partial \psi_m}{\partial \eta_n} \\ (i = 1, 2, \cdots, 6), \end{cases}$$

$$X_i = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} \sum_{j=1}^{j=6} B_{ke} X^{(k)} \mu_{ij} \frac{\partial \eta_j}{\partial q_e} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} \mathfrak{X}(\psi_m) \frac{\partial \psi_e}{\partial \eta_i}, \tag{20}$$

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} X_i X_j = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} X^{(k)} X^{(e)} + \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{X}(\psi_k) \mathfrak{X}(\psi_e). \tag{21}$$

The last formula will be required in the next chapter.

CHAPTER VII.

Return to the Perturbative Function of Convective Motion.

§1. *The function* G.—Let us again take up the expression:

$$G_2 = G - \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} h_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'}, \tag{1}$$

and let us now consider the case of motion of a solid body subject to s conditions of constraint as in the preceding chapter. It will be observed that in this case the coefficients h_{ke} which figure in (1) are identical with the coefficients B_{ke} introduced in chapter VI, so that now

$$G_2 = G - \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'}. \tag{1'}$$

This expression will be transformed by means of the formulas of chapter VI. For this purpose put $X_i = \partial L / \partial \eta_i'$. Then

$$X^{(k)} = \sum_{i=1}^{i=6} \frac{\partial L}{\partial \eta_i'} \frac{\partial \eta_i}{\partial q_k} = \frac{\partial L}{\partial q_k'},$$

and therefore, by formula (21) of the preceding chapter,

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \frac{\partial L}{\partial \eta_i'} \frac{\partial L}{\partial \eta_j'} = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'} + \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} L(\psi_k) L(\psi_e),$$

where

$$L(\psi_k) = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \frac{\partial L}{\partial \eta_i'} \frac{\partial \psi_k}{\partial \eta_j}. \tag{2}$$

But we have found in §6, chapter V, that

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \frac{\partial L}{\partial \eta_i'} \frac{\partial L}{\partial \eta_j'} = 2G;$$

hence,

$$G = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'} + \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} L(\psi_k) L(\psi_e). \tag{3}$$

§2. *Perturbative function for a free solid body.*—Substituting the expression (3 of G into formula (1′) we find that, for a solid body,

$$G_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} L(\psi_k) L(\psi_e); \tag{4}$$

i. e., the function G_2 is due solely to the existence of constraints.

If $s = 0$, i. e., if the body is free, then $G_2 = 0$. We thus find again the theorem of §7, chapter V, namely, that *for a free solid body the perturbative function of convective motion reduces to* $-(K + L)$.

§3. *Cases when $G_2 = 0$.*—We have already seen in chapter V that by extending in a familiar way the results obtained for rigid systems of particles to solid bodies we shall have:

$$L(\psi_k) = pR_x^{(k)} + qR_y^{(k)} + rR_z^{(k)} = \omega R^{(k)} \cos(\omega, R^{(k)}), \tag{5}$$

where $R^{(k)}$ is a vector parallel and proportional to the moment of reaction relatively to the origin of the axes XYZ due to the constraint ψ_k. Formula (5) shows that $L(\psi_k)$ *vanishes whenever this moment vanishes or when its direction forms a right angle with the direction of the instantaneous axis* (ω).

Two interesting cases when all the $L(\psi_k)$ vanish may be mentioned.

1. The body can freely rotate about the origin of the axes XYZ.

2. The body can freely rotate about a point which constantly remains on the instantaneous axis (ω).

In fact, in the first case the moments of the constraints vanish; in the second, their directions form right angles with the direction of the instantaneous axis (ω). Hence this theorem:

When a solid body can rotate freely about the origin of the axes XYZ, or about a point constantly remaining on the instantaneous axis (ω)*; then the perturbative function of convective motion reduces to* $-(K + L)$.

§4. *The perturbative function for very small values of* ω. Again, it will be observed as in chapter V that the perturbative function is the sum of the three functions $-K$, $-L$, $-G_2$, all three homogeneous in p, q, r; the degree of homogeneity being 0 for K, 1 for L, 2 for G_2.

If we look upon ω as a small quantity of the first order and upon Ω as a function of ω, we can say that *the perturbative function of the second order in* ω *is due solely to the existence of constraints and vanishes when the body is free, or when it freely rotates about the origin of the axes of relative coördinates, or finally, when it freely rotates about a point constantly remaining on the instantaneous axis* (ω).

CHAPTER VIII.

Decomposition of the Kinetic Energy of a Material System. Mechanical Significance of the Function G_2.

§1. *On the nature of constraints.*—In order that certain theorems enunciated below may be properly understood, a few words as to the manner of interpreting geometrical conditions are necessary.

Suppose that a system of n particles be subject in its motion to the condition:*

$$(1) \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, x_i, y_i, z_i, x_{i+1}, \cdots, x_n, y_n, z_n) = 0,$$

not involving the time explicitly. We know that, geometrically interpreted, this condition involves the following:—

1) A particle m_i (x_i, y_i, z_i) remains on the surface:

$$(2) \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, x_{i+1}, \cdots, x_n, y_n, z_n) = 0,$$

whose form and position in space depend on the $3(n-1)$ parameters $x_1, y_1, \cdots, z_{i-1}, x_{i+1}, \cdots, z_n$.

2) The velocity of this particle is tangential to the surface (2).

These conditions can be expressed in a somewhat different form.

1) A particle m_i must remain on the surface (2). This surface changes its form and position in space during the motion.

2) The variation in form and position of the surface (2) imparts to each of its points a certain velocity v whose normal and tangential components will be denoted by v_N and v_T.

3) When the equations of constraint, as in (1), do not involve the time explicitly, then the normal velocity v_N at the point coinciding with the particle m_i is equal in magnitude and direction to the normal component of the velocity of this particle. In consequence of this the velocity of the particle m_i will be tangential to the surface (2).

4) The tangential velocity v_T is without effect on the particle m_i, the surface (2) being *ideal*, *i. e.*, void of friction.

Now, suppose that the given material system is subject to the condition:

$$(1') \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, x_i, y_i, z_i, x_{i+1}, \cdots, x_n, y_n, z_n, t) = 0,$$

involving the time explicitly. Geometrically this condition may be interpreted as follows:—

* Unless otherwise stated, x, y, z in this chapter denote coördinates with regard to any system of axes, whether fixed in space or moving.

1) A particle m_i must remain on the surface:

$$(2') \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, x_{i+1}, \cdots, x_n, y_n, z_n, t) = 0,$$

whose form and position in space depend on the $3(n-1)+1$ parameters x_1, $\cdots$, z_{i-1}, x_{i+1}, $\cdots$, z_n, t.

2) The variation in form and position of the surface (2) imparts to each of its points a certain velocity v whose normal and tangential components are v_N and v_T.

3) When the equations of constraint, as in (1'), involve the time explicitly, then the velocity v_N, at the point coinciding with the particle m_i, is no longer equal to the normal component of the velocity of this particle. In consequence of this the surface (2') will exercise an *impulse* on the particle m_i, normal to the surface (2), and whose components along the axes XYZ will therefore be of the form $\lambda\,\partial\phi/\partial x_i$, $\lambda\,\partial\phi/\partial y_i$, $\lambda\,\partial\phi/\partial z_i$.

4) The tangential component of the velocity of the particle m_i will satisfy the given condition (2') if the parameter t be considered constant; or, to be more precise, if w_i' denote the tangential component of the velocity w_i of a particle m_i, we shall have:

$$0 = \sum_i \left(w'_{ix} \frac{\partial\phi}{\partial x_i} + w'_{iy} \frac{\partial\phi}{\partial y_i} + w'_{iz} \frac{\partial\phi}{\partial z_i} \right).$$

5) The tangential velocity v_T, again, is without effect on the particle m_i.

If, instead of the sole condition (1'), we had several such conditions involving the time explicitly, it is clear that the several surfaces $\phi_1 = 0$, $\phi_2 = 0$, $\cdots$, would each exercise a normal impulse on the particle; so that, $m_i(w_i)_2$ denoting the *resultant* of all these impulses, we may write

$$(2) \qquad \begin{cases} m_i(w_{ix})_2 = \sum_k \lambda_k \dfrac{\partial\phi_k}{\partial x_i}, \\ m_i(w_{iy})_2 = \sum_k \lambda_k \dfrac{\partial\phi_k}{\partial y_i}, \\ m_i(w_{iz})_2 = \sum_k \lambda_k \dfrac{\partial\phi_k}{\partial z_i}. \end{cases}$$

Now, $(w_i)_2$ being a component of the actual velocity w_i of the particle m_i, let $(w_i)_1$ be the other component. It is clear from the above that the velocities $(w_i)_1$ will comply with the given conditions of constraint if the time, so far as it enters *explicitly* into their equations, be considered constant, i. e., we shall have:

$$(4) \qquad 0 = \sum_i \left\{ (w_{ix})_1 \frac{\partial\phi_k}{\partial x_i} + (w_{iy})_1 \frac{\partial\phi_k}{\partial y_i} + (w_{iz})_1 \frac{\partial\phi_k}{\partial z_i} \right\} \qquad (k = 1, 2, \cdots).$$

§2. *Decomposition of the kinetic energy of a material system.*—It follows from formulas (3) and (4) that

$$0 = \sum_{i=1}^{i=n} m_i \{(w_{ix})_1 (w_{ix})_2 + (w_{iy})_1 (w_{iy})_2 + (w_{iz})_1 (w_{iz})_2\}; \tag{5}$$

and therefore, if we put

$$\mathfrak{T} = \tfrac{1}{2} \sum_{i=1}^{i=n} m_i w_i^2, \quad \mathfrak{T}_1 = \tfrac{1}{2} \sum_{i=1}^{i=n} m_i (w_i)_1^2, \quad \mathfrak{T}_2 = \tfrac{1}{2} \sum_{i=1}^{i=n} m_i (w_i)_2^2, \tag{6}$$

we shall have:

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2, \tag{7}$$

which expresses the following theorem:

The kinetic energy $\mathfrak{T}$ of a system of particles subject to conditions of constraint involving the time explicitly is the sum of the kinetic energy $\mathfrak{T}_1$ which the given material system would have if at the considered moment the time, so far as it enters explicitly in the equations of constraint, became constant; and of the kinetic energy $\mathfrak{T}_2$ which is imparted to the material system by the variation of the time, so far as it enters explicitly in these equations.

§3. *Expressions of* $(w_i)_1$, $(w_i)_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$.—To determine these put

$$W^{(k)} = \sum_{i=1}^{i=n} m_i \left(w_{ix} \frac{\partial x_i}{\partial q_k} + w_{iy} \frac{\partial y_i}{\partial q_k} + w_{iz} \frac{\partial z_i}{\partial q_k} \right), \tag{8}$$

$q_1, q_2, \cdots, q_\mu$ being the independent Lagrangian coördinates of the system, and

$$W(\phi_k) = \sum_{i=1}^{i=n} \left(w_{ix} \frac{\partial \phi_k}{\partial x_i} + w_{iy} \frac{\partial \phi_k}{\partial y_i} + w_{iz} \frac{\partial \phi_k}{\partial z_i} \right). \tag{9}$$

It is readily seen with the help of formulas (3) and (4) that

$$W(\phi_k) = \sum_{i=1}^{i=n} \left\{ (w_{ix})_2 \frac{\partial \phi_k}{\partial x_i} + (w_{iy})_2 \frac{\partial \phi_k}{\partial y_i} + (w_{iz})_2 \frac{\partial \phi_k}{\partial z_i} \right\} = \sum_{e=1}^{e=s} \lambda_e \phi_{ke} \tag{10}$$

$(k = 1, 2, \cdots, s)$,

ϕ_{ke} being defined by (d), chap. IV. The s equations (10) being solved with regard to $\lambda_1, \cdots, \lambda_s$ give

$$\lambda_k = \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_e) \qquad (k = 1, 2, \cdots, s). \tag{11}$$

Substituting these expressions of the λ_k in (3) we obtain:

$$(12)\qquad \begin{cases} m_i(w_{ix})_2 = \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \dfrac{\partial \phi_e}{\partial x_i}, \\ m_i(w_{iy})_2 = \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \dfrac{\partial \phi_e}{\partial y_i}, \\ m_i(w_{iz})_2 = \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \dfrac{\partial \phi_e}{\partial z_i}. \end{cases}$$

To determine the (w_i), observe that the conditions (4) will be satisfied by selecting the velocities (w_i) so that

$$(13)\qquad \begin{cases} (w_{ix})_1 = \sum_{k=1}^{k=\mu} \mu_k \dfrac{\partial x_i}{\partial q_k}, \\ (w_{iy})_1 = \sum_{k=1}^{k=\mu} \mu_k \dfrac{\partial y_i}{\partial q_k}, \\ (w_{iz})_1 = \sum_{k=1}^{k=\mu} \mu_k \dfrac{\partial z_i}{\partial q_k}. \end{cases}$$

It remains therefore to determine the μ_k. To this end I remark that

$$\sum_{i=1}^{i=n} m_i \left\{ (w_{ix})_2 \frac{\partial x_i}{\partial q_k} + (w_{iy})_2 \frac{\partial y_i}{\partial q_k} + (w_{iz})_2 \frac{\partial z_i}{\partial q_k} \right\} = \sum_{e=1}^{e=s} \lambda_e \frac{\partial \phi_e}{\partial q_k} \equiv 0 \qquad (k=1, 2, \cdots, \mu),$$

and consequently,

$$(14)\quad W^{(k)} = \sum_{i=1}^{i=n} m_i \left\{ (W_{ix})_1 \frac{\partial x_i}{\partial q_k} + (w_{iy})_1 \frac{\partial y_i}{\partial q_k} + (w_{iz})_1 \frac{\partial z_i}{\partial q_k} \right\} = \sum_{e=1}^{e=\mu} \mu_e a_{ke} \qquad (k=1, 2, \cdots, \mu),$$

a_{ke} being defined as in §3, chap. IV. From the μ equations (14) are readily derived the following values of $\mu_1, \cdots, \mu_\mu$:

$$(15)\qquad \mu_k = \sum_{e=1}^{e=\mu} A_{ke} W^{(e)} \qquad (k=1, 2, \cdots, \mu).$$

Substituting these expressions of the μ_k in (13) we obtain:

$$(16)\qquad \begin{cases} (w_{ix})_1 = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \dfrac{\partial x_i}{\partial q_e}, \\ (w_{iy})_1 = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \dfrac{\partial y_i}{\partial q_e}, \\ (w_{iz})_1 = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \dfrac{\partial z_i}{\partial q_e}. \end{cases}$$

By means of (16) and (12) we find without difficulty that

$$\mathfrak{T}_1 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} W^{(e)}, \tag{17}$$

$$\mathfrak{T}_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) W(\phi_e). \tag{18}$$

§4. *A special case.*—In particular let

$$w_{ix} = \frac{dx_i}{dt} = x_i', \quad w_{iy} = y_i', \quad w_{iz} = z_i'.$$

Then

$$\mathfrak{T} = T, \quad W^{(k)} = \frac{\partial T}{\partial q_k'} = p_k, \quad W(\phi_k) = -\frac{\partial \phi_k}{\partial t};$$

and the results of §6, chapter IV are obtained in a new form which lends itself more readily to a mechanical interpretation. Put

$$u_i' = (u_i')_1 + (u_i')_2, \tag{19}$$

the u_i being defined as in chapter IV, and we shall then have:

$$(u_i')_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial u_i}{\partial q_e} p_k, \tag{20}$$

$$(u_i')_2 = -\sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_e}{\partial t} \frac{\partial \phi_k}{\partial u_i}, \tag{21}$$

simultaneously with the formulas:

$$T = T_1 + T_2, \tag{22}$$

$$T_1 = \tfrac{1}{2} \sum_{i=1}^{i=3n} (u_i')_1^2 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} p_k p_e, \tag{23}$$

$$T_2 = \tfrac{1}{2} \sum_{i=1}^{i=3n} (u_i')_2^2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_k}{\partial t} \frac{\partial \phi_e}{\partial t}. \tag{24}$$

The mechanical interpretation of these formulas follows at once from the theorem enunciated in §2. Indeed, nothing need be changed in the enunciation of that theorem in order to apply it in the present case, except the letter $\mathfrak{T}$ into the letter T.

§5. *New expressions for formulas* (2) *and* (3) *of chapter IV.*—Another demonstration of the theorem of §2, based on the formulas of chapter IV, will now be given. It was shown in §5 of that chapter that we may write

$$X_i = (X_i)_1 + (X_i)_2, \tag{25}$$

where

$$(X_i)_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} X^{(k)} \frac{\partial u_i}{\partial q_e}, \tag{26}$$

$$(X_i)_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke}(\phi_k) \frac{\partial \phi_e}{\partial u_i}. \tag{27}$$

These formulas show that

$$\sum_{i=1}^{i=3n} (X_i)_1 \frac{\partial \phi_k}{\partial u_i} = 0 \qquad (k = 1, 2, \cdots, s), \tag{28}$$

and subsequently, that

$$\sum_{i=1}^{i=3n} (X_i)_1 (X_i)_2 = 0. \tag{29}$$

Therefore

$$\sum_{i=1}^{i=3n} (X_i)^2 = \sum_{i=1}^{i=3n} (X_i)_1^2 + \sum_{i=1}^{i=3n} (X_i)_2^2, \tag{30}$$

with

$$\sum_{i=1}^{i=3n} (X_i)_1^2 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} X^{(k)} X^{(e)}, \tag{31}$$

$$\sum_{i=1}^{i=3n} (X_i)_2^2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} X(\phi_k) X(\phi_e). \tag{32}$$

§6. *Another proof of the theorem of* §2.—Let w_i be the velocity of a particle and let $X_1, X_2, \cdots, X_{3n}$ be respectively $\sqrt{m_1}\, w_{1x}, \sqrt{m_1}\, w_{1y}, \cdots, \sqrt{m_n}\, w_{nz}$; let moreover $\mathfrak{T}$, $W^{(k)}$, $W(\phi_k)$ be defined as before by (6), (8), (9). Applying, first, formula (25) of the preceding paragraph, we find:

$$\begin{cases} w_{ix} = (w_{ix})_1 + (w_{ix})_2, \\ w_{iy} = (w_{iy})_1 + (w_{iy})_2, \\ w_{iz} = (w_{iz})_1 + (w_{iz})_2; \end{cases} \tag{33}$$

where, according to (25) and (27),

$$\begin{cases} (w_{ix})_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial x_i}{\partial q_e}, \\ (w_{iy})_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial y_i}{\partial q_e}, \\ (w_{iz})_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial z_i}{\partial q_e}; \end{cases} \tag{34}$$

$$\begin{cases} m_i (w_{ix})_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \frac{\partial \phi_e}{\partial x_i}, \\ m_i (w_{iy})_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \frac{\partial \phi_e}{\partial y_i}, \\ m_i (w_{iz})_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \frac{\partial \phi_e}{\partial z_i}. \end{cases} \tag{35}$$

At the same time, by (30),

$$\sum_{i=1}^{i=n} m_i w_i^2 = \sum_{i=1}^{i=n} m_i (w_i)_1^2 + \sum_{i=1}^{i=n} m_i (w_i)_2^2,$$

or, in the notation (6),

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2$$

which is the formula (7) of §2 and expresses the theorem there enunciated.

Formulas (34) and (35) are identical with (16) and (12). The proof of these formulas and of the theorem of §2, as given in this paragraph, is possibly more rigorous than the demonstration previously given. The advantage of the other method lies in its immediate geometrical interpretation. For these reasons it seemed expedient to give both proofs.

§ 7. *Specific virtual velocities. Constraint velocities.*—For the convenience of formulating the results two new terms will now be introduced. Referring to the equation (4) it is readily seen that the velocities $(w_i)_1$ belong to the class of so-called *virtual velocities*, a term used in deriving D'Alembert's principle. The velocities $(w_i)_1$ however are *perfectly determinate* at each moment, while in general to each particle m_i, at a given moment, correspond an infinity of virtual velocities. Among the latter is the component $(w_i)_1$ of the actual velocity w_i. I call $(w_i)_1$ for lack of a better term the *specific virtual velocity* of the particle m_i.

The term proposed for the other component of w_i is justified by the geometrical interpretation given in the beginning of this chapter. I call $(w_i)_2$ the *constraint velocity* of the particle m_i.

The theorem of §2 will now read as follows:

The kinetic energy of a system of particles subject to conditions of constraint involving the time explicitly is the sum of the kinetic energy of the material system arising from specific virtual velocities and of the kinetic energy due to the constraint velocities of the particles.

§ 8. *Kinetic energy imparted to a material system by convection.*—Suppose that the kinetic energy $\mathfrak{T}$ is imparted to the material system through a motion of the axes XYZ. Suppose moreover that the equations of constraint, when expressed in terms of the relative coördinates of the particles, do not involve the time explicitly. Then the velocities $(w_i)_1$ will satisfy the same conditions as the *actual relative velocities* of the particles; in other words, it would be *possible* for a particle m_i to assume the velocity $(w_i)_1$ in its *relative* motion. On the other hand, the velocities $(w_i)_2$ being resultants of velocities normal to surfaces to which the relative velocities are tangential,* it is clear that it would be *impos-*

* On account of the condition that the equations of constraint expressed in terms of the relative coördinates do not involve the time explicitly.

sible for a particle m_i to assume the velocity $(w_i)_2$ in its *relative* motion. Hence the following theorem:

The convective kinetic energy of a system of particles, when the conditions of constraint expressed in the relative coördinates do not involve the time explicitly, is the sum of the kinetic energy arising from certain velocities which it is possible for the particles to assume in their relative motion, and of the kinetic energy due to certain other velocities which it would be impossible for the particles to assume in their relative motion.†*

§9. *Mechanical interpretation of the function* G_2.—The results obtained in the preceding paragraphs will now be applied to the perturbative function of convective motion.

Let again x_i, y_i, z_i denote the relative coördinates of a particle and let

$$w_{ix} = qz_i - ry_i, \quad w_{iy} = rx_i - pz_i, \quad w_{iz} = py_i - qx_i. \tag{36}$$

Then

$$\mathfrak{T} = G,$$

and, since now

$$m_i w_{ix} = \frac{\partial L}{\partial x_i'}, \quad m_i w_{iy} = \frac{\partial L}{\partial y_i'}, \quad m_i w_{iz} = \frac{\partial L}{\partial z_i'},$$

we shall have in the present case:

$$W^{(k)} = \sum_{i=1}^{i=3n} \frac{\partial L}{\partial u_i'} \frac{\partial u_i}{\partial q_k} = \frac{\partial L}{\partial q_k'},$$

$$W(\phi_k) = \sum_{i=1}^{i=3n} \frac{\partial L}{\partial u_i'} \frac{\partial \phi_k}{\partial u_i} = L(\phi_k);$$

and the formulas of §§2, 3 yield the following:

$$G = G_1 + G_2, \tag{37}$$

$$G_1 = \tfrac{1}{2} \sum_{i=1}^{i=n} m_i (w_i)_1^2 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'}, \tag{38}$$

$$G_2 = \tfrac{1}{2} \sum_{i=1}^{i=n} m_i (w_i)_2^2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} L(\phi_k) L(\phi_e), \tag{39}$$

which go to show that G_2, as given by the decomposition of the kinetic energy G, is nothing else than the third term of the perturbative function of convective motion, so designated before.

* i. e., the kinetic energy imparted to the material system by a motion of the axes *XYZ*.

† While this theorem, in general, involves an element of indeterminateness, namely the manifoldness of possible and of impossible velocities, in the applications this indeterminateness often vanishes; thus when a particle can move in only one direction, or, on the contrary, when it can move in every direction but one, the problem of decomposing the kinetic energy is obviously reduced. See as an illustration the example given in §13, chap. IX.

Before interpreting these results I make the following remark:—

Let $\phi(x_1, y_1, z_1, \cdots, z_n, t) = 0$ be a condition of constraint expressed in terms of the relative coördinates of the particles and let the function ϕ become $f(\xi_1, \eta_1, \zeta_1, \cdots, \zeta_n, t)$ when the absolute coördinates are introduced instead of the relative. We consider, as in §1, the two surfaces:

$$\phi(x_1, y_1, z_1, \cdots, z_{i-1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, x_{i+1}, \cdots, t) = 0, \tag{40}$$

$$f(\xi_1, \eta_1, \zeta_1, \cdots, \zeta_{i-1}, \Xi, \mathrm{H}, \mathrm{Z}, \xi_{i+1}, \cdots, t) = 0. \tag{40'}$$

and we find that the variation of the surface (40) in form and position relatively to the axes XYZ imparts to a particle m_i an impulse $m_i(v_i)_2^{(r)}$; the variation of the surface (40′) in form and position in absolute space imparts to the same particle an impulse $m_i(v_i)_2^{(a)}$. Now, at each moment t, the equations (40) and (40′) represent the same surface. Hence, the variation of this surface in absolute space is equivalent to the combined, (1) variation of the same surface relatively to the axes XYZ, and (2) its convective motion with the axes XYZ. The first imparts to the particle m_i the impulse $m_i(v_i)_2^{(r)}$; the second imparts to it an impulse $m_i(v_i)_2^{(c)}$; and it is clear that $(v_i)_2^{(a)}$ will be the resultant of velocities $(v_i)_2^{(r)}$ and $(v_i)_2^{(c)}$. In the case of more than one condition of constraint a generalization of the preceding result is readily obtained. We may say therefore that

The absolute constraint velocity of a particle is the resultant of its relative and its convective constraint velocities.

A similar proposition holds for the specific virtual velocity of a particle.

In particular, if the equations of constraint expressed in terms of the relative coördinates of the particles do not involve the time explicitly, then the relative constraint velocities vanish and therefore the constraint velocity of a particle is due only to convection.

Now, if we observe that the expressions (36) are the components along the axes XYZ of the convective rotary velocity of a particle, we may interpret formula (39) as follows:—

G_2 is the kinetic energy of the material system due to constraint velocities arising from convective rotation.

Note.—It may be well to supplement the geometrical interpretation given above by the following remark which is suggested by HANSEN'S method of computing perturbations.

The transformation of coördinates:

$$(a) \qquad \begin{cases} x_i = a_i + a_{i1}\xi_i + \beta_{i1}\eta_i + \gamma_{i1}\zeta_i, \\ y_i = b_i + a_{i2}\xi_i + \beta_{i2}\eta_i + \gamma_{i2}\zeta_i, \\ z_i = c_i + a_{i3}\xi_i + \beta_{i3}\eta_i + \gamma_{i3}\zeta_i; \end{cases}$$

may be effected by the successive steps

$$(b)\qquad \begin{cases} x_i = a_i + \bar{x}_i, \\ y_i = b_i + \bar{y}_i, \\ z_i = c_i + \bar{z}_i; \end{cases}$$

$$(c)\qquad \begin{cases} \bar{x}_i = \alpha_{i1}\xi_i + \beta_{i1}\eta_i + \gamma_{i1}\zeta_i, \\ \bar{y}_i = \alpha_{i2}\xi_i + \beta_{i2}\eta_i + \gamma_{i2}\zeta_i, \\ \bar{z}_i = \alpha_{i3}\xi_i + \beta_{i3}\eta_i + \gamma_{3i}\zeta_i. \end{cases}$$

The several coefficients entering into these formulas are functions of the time. A distinct letter for the time in each case will be used, namely $t, \bar{\tau}, \tau$.

Now, let

$$(d)\qquad \phi(x_1, y_1, z_1, \cdots, x_n, y_n, z_n, t) = 0$$

be the equation of a constraint expressed in terms of the relative coördinates. Introducing the absolute coördinates by means of (a), equation (d) takes the form:

$$(e)\qquad f(\xi_1, \eta_1, \zeta_1, \cdots, \xi_n, \eta_n, \xi_n, t) = 0.$$

Introducing the same coördinates into (d) by the successive transformations (b) and (c) we first obtain,

$$F(\bar{x}_1, \bar{y}_1, \bar{z}_1, \cdots, \bar{x}_n, \bar{y}_n, \bar{z}_n, \bar{\tau}, t) = 0,$$

and next,

$$(f)\qquad \Phi(\xi_1, \eta_1, \zeta_1, \cdots, \xi_n, \eta_n, \zeta_n, \tau, \bar{\tau}, t) = 0.$$

The equations (e) and (f) represent the same constraint.

Now it is clear, that by varying the time in (f) only *as far as it enters explicitly and in the form of the letter* t we shall impart to a particle the same constraint velocity as the corresponding variation of the time, so far as it enters explicitly in the equation (d), would impart to this particle. This constraint velocity was termed above the *relative constraint velocity* of the particle. Similarly, the variation of the time in (f), *as far as it appears under the letter* $\bar{\tau}$, will impart to a particle the *constraint velocity of convective translation;* and finally, the variation of the time, *as far as it enters in* (f) *in the form of the letter* τ, will impart to a particle the *constraint velocity of convective rotation.*

The resultant of these velocities is the *absolute constraint velocity* of the particle. This resultant would be imparted to the particle directly, by the corresponding variation of the time, as far as it enters explicitly in the equation (e).

On the other hand, by assuming that at a given moment the time under the respective letters t, τ, τ became constant in (f), we should arrive at the notion

of the *relative specific virtual velocity* of a particle and of the *specific virtual velocities of convective translation and of rotation* respectively. We should also find that the resultant of these velocities is the *absolute specific virtual velocity* of the particle, i. e., the absolute velocity which this particle would assume if the time, as far as it enters explicitly in equation (e), became constant at the considered moment.

CHAPTER IX.

Decomposition of the Kinetic Energy of a Solid Body. Mechanical Interpretation of the Function G_2.

§1. *Specific virtual velocity and constraint velocity of a solid body.*—Let v_i be the velocity of the body along the parameter η_i and let $\mathfrak{T}$ denote the kinetic energy of the body due to these velocities, i. e., let

$$\mathfrak{T} = \tfrac{1}{2} \sum \sum \mu_{ij} v_i v_j . \tag{1}$$

I remark, as in the preceding chapter, that the velocity v_i of the body along the parameter η_i may be considered as the resultant of the velocity $(v_i)_1$ which the body would have along the same parameter if at the considered moment the time, as far as it enters *explicitly* into the equations of constraint, became constant; and of the velocity $(v_i)_2$ imparted to the body along the parameter η_i by the variation of the time as far as it enters *explicitly* in these equations. I shall call $(v_i)_1$ the *specific virtual velocity* and $(v_i)_2$ the *constraint velocity* of the body along the parameter η_i.*

It may be remarked, as in the preceding chapter, that the velocities $(v_i)_1$ satisfy the conditions:

$$0 = \sum_{i=1}^{i=6} (v_i)_1 \frac{\partial \psi_k}{\partial \eta_i} \qquad (k = 1, 2, \cdots, s), \tag{2}$$

while the impulses $(\partial \mathfrak{T}/\partial v_i)_2$ imparted to the body by the variation of the time, as far as it enters explicitly in the equations of constraint, must have the form $\sum \lambda_k \partial \psi_k / \partial \eta_i$, so that we can write:

$$\sum_{j=1}^{j=6} \mu_{ij} (v_j)_2 = \left(\frac{\partial \mathfrak{T}}{\partial v_i} \right)_2 = \sum_{k=1}^{k=s} \lambda_k \frac{\partial \psi_k}{\partial \eta_i} \qquad (i = 1, 2, \cdots, 6). \tag{3}$$

* Unless explicitly stated otherwise the parameters $\eta_1, \cdots, \eta_6$ define the position of the body with regard to any system of axes, fixed in space or moving.

§2. *Decomposition of the kinetic energy of a solid body.*—It follows from (2) and (3) that

$$0 = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij}(v_i)_1(v_j)_2, \tag{4}$$

and consequently,

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} v_i v_j = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij}(v_i)_1(v_j)_1 + \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij}(v_i)_2(v_j)_2;$$

or, if we put

$$\mathfrak{T}_1 = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij}(v_i)_1(v_j)_1, \quad \mathfrak{T}_2 = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij}(v_i)_2(v_j)_2, \tag{5}$$

and take into account formula (1),

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2. \tag{6}$$

We have thus arrived at the following theorem:

The kinetic energy of a solid body, subject to conditions of constraint involving the time explicitly, is the sum of the kinetic energy of the body arising from the specific virtual velocities and of the kinetic energy due to the constraint velocities.

§3. *Expressions of* $(v_i)_1$, $(v_i)_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$.—To determine these, put

$$\mathfrak{V}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} v_i \frac{\partial \eta_j}{\partial q_k}, \tag{7}$$

$$\mathfrak{V}(\psi_k) = \sum_{i=1}^{i=6} v_i \frac{\partial \psi_k}{\partial \eta_i}, \tag{8}$$

$q_1, q_2, \cdots$ being the independent Lagrangian coördinates of the body.

It is readily seen, by means of (2) and (3) that *

$$\mathfrak{V}(\psi_k) = \sum_{i=1}^{i=6} (v_i)_2 \frac{\partial \psi_k}{\partial \eta_i} = \sum_{e=1}^{e=s} \lambda_e \psi_{ek} \qquad (k = 1, 2, \cdots, s).$$

From these s equations we derive:

$$\lambda_k = \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{V}(\psi_e) \qquad (k = 1, 2, \cdots, s).$$

* The six equations (3) solved with respect to $(v_1)_2, \cdots, (v_6)_2$ give

$$(v_i)_2 = \sum_{k=1}^{k=s} \sum_{j=1}^{j=6} \lambda_k M_{ij} \frac{\partial \psi_k}{\partial \eta_j} \qquad (i = 1, 2, \cdots, 6). \tag{3'}$$

Substituting these expressions of the λ_k in (3) and (3′) we obtain:

$$\left(\frac{\partial \mathfrak{T}}{\partial v_i}\right)_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{V}(\psi_k) \frac{\partial \psi_e}{\partial \eta_i} \qquad (i = 1, 2, \cdots, 6), \tag{9}$$

$$(v_i)_2 = \sum_{j=1}^{j=6} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{V}(\psi_k) M_{ij} \frac{\partial \psi_e}{\partial \eta_j} \qquad (i = 1, 2, \cdots, 6). \tag{10}$$

To determine the velocities $(v_i)_1$ I observe that the conditions (2) will be satisfied by giving to $(v_i)_1$ the form:

$$(v_i)_1 = \sum_{k=1}^{k=\mu} \mu_k \frac{\partial \eta_i}{\partial q_k}. \tag{11}$$

It remains therefore to determine the μ_k. For this we have:

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (v_i)_2 \frac{\partial \eta_j}{\partial q_k} = \sum_{e=1}^{e=s} \lambda_e \frac{\partial \psi_e}{\partial q_k} \equiv 0 \qquad (k = 1, 2, \cdots, \mu),$$

and consequently,

$$\mathfrak{V}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (v_i)_1 \frac{\partial \eta_j}{\partial q_k} = \sum_{e=1}^{e=s} \mu_e b_{ke} \qquad (k = 1, 2, \cdots, \mu),$$

b_{ke} being defined in §3, chap. VI. From the last μ equations we derive

$$\mu_k = \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{V}^{(e)} \qquad (k = 1, 2, \cdots, \mu);$$

and substituting these expressions of the μ_k in (11), we obtain:

$$(v_i)_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{V}^{(k)} \frac{\partial \eta_i}{\partial q_e} \qquad (i = 1, 2, \cdots, 6). \tag{12}$$

By means of (9) and (12) we now find without difficulty that

$$\mathfrak{T}_1 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{V}^{(k)} \mathfrak{V}^{(e)}, \tag{13}$$

$$\mathfrak{T}_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{V}(\psi_k) \mathfrak{V}(\psi_e). \tag{14}$$

§4. *A special case.*—In particular, let $v_i = d\eta_i/dt = \eta_i'$. Then

$$\mathfrak{T} = T, \quad \mathfrak{V}^{(k)} = \frac{\partial T}{\partial q_k'} = p_k, \quad \mathfrak{V}(\psi_k) = -\frac{\partial \psi_k}{\partial t};$$

and the results of §6, chap. VI, are obtained in a new form which lends itself more readily to a mechanical interpretation. Put

$$\eta_i' = (\eta_i')_1 + (\eta_i')_2, \tag{15}$$

and we shall then have:

$$(\eta_i')_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \frac{\partial \eta_i}{\partial q_e} p_k, \tag{16}$$

$$(\eta_i')_2 = -\sum_{n=1}^{n=6} \sum_{e=1}^{e=s} \sum_{k=1}^{k=s} \Psi_{ke} M_{in} \frac{\partial \psi_e}{\partial t} \frac{\partial \psi_k}{\partial \eta_n}, \tag{17}$$

simultaneously with the formulas:

$$T = T_1 + T_2, \tag{18}$$

$$T_1 = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\eta_i')_1 (\eta_j')_1 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} p_k p_e, \tag{19}$$

$$T_2 = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\eta_i')_2 (\eta_j')_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \frac{\partial \psi_k}{\partial t} \frac{\partial \psi_e}{\partial t}. \tag{20}$$

The mechanical interpretation of these formulas follows immediately from the theorem enunciated in §2; in fact the only change in its formulation necessary to apply it in the present case is the substitution of T for $\mathfrak{T}$.

§5. *New expressions for formulas* (13) *and* (15) *of chap. VI.* As in the preceding chapter another proof of the theorem of §2 may be given, on the strength of the formulas of chap. VI; although the present demonstration does not yield a mechanical interpretation of the results as readily as the one previously given, it has the advantage of being more rigorous.

We have seen in chap. VI, that in the case of a solid body we may write:

$$X_i = (X_i)_1 + (X_i)_2, \tag{21}$$

where

$$(X_i)_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} X^{(e)} \frac{\partial \eta_i}{\partial q_k}, \tag{22}$$

$$(X_i)_2 = \sum_{n=1}^{n=6} \sum_{e=1}^{e=s} \sum_{k=1}^{k=s} \Psi_{ke} M_{in} X(\psi_k) \frac{\partial \psi_i}{\partial \eta_n}. \tag{23}$$

These expressions show that

$$\sum_{i=1}^{i=6} (X_i)_1 \frac{\partial \psi_k}{\partial \eta_i} = 0, \qquad (k = 1, 2, \cdots, s), \tag{24}$$

and consequently,

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (X_i)_1 (X_j)_2 = 0. \tag{25}$$

Therefore

$$(26)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}X_iX_j=\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_1(X_j)_1+\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_2(X_j)_2,$$

and

$$(27)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_1(X_j)_1=\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}\mathfrak{X}^{(k)}\mathfrak{X}^{(e)},$$

$$(28)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_2(X_j)_2=\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}X(\psi_k)X(\psi_e).$$

§ 6. *Another proof of the theorem of* §2.—Let v_i again denote the velocity of the body along the parameter η_i and let us put $X_i=v_i$; let also $\mathfrak{T}$, $\mathfrak{V}^{(k)}$, $\mathfrak{V}(\psi_k)$ be defined, as before, by (1), (7), (8). Applying formula (21) of the preceding paragraph, we shall have:

$$(29)\qquad v_i=(v_i)_1+(v_i)_2,$$

where, according to (22) and (23),

$$(30)\qquad (v_i)_1=\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}\mathfrak{V}^{(e)}\frac{\partial\eta_i}{\partial q_k},$$

$$(31)\qquad (v_i)_2=\sum_{n=1}^{n=6}\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}M_{in}\mathfrak{V}(\psi_k)\frac{\partial\psi_e}{\partial\eta_n}.$$

At the same time, by (26),

$$\sum\sum\mu_{ij}v_iv_j=\sum\sum\mu_{ij}(v_i)_1(v_j)_1+\sum\sum\mu_{ij}(v_i)_2(v_j)_2,$$

or, in the notation (1) and (5),

$$\mathfrak{T}=\mathfrak{T}_1+\mathfrak{T}_2,$$

which is the formula (6) of § 2 and expresses the theorem as there enunciated.

Formulas (30) and (31) are identical with formulas (12) and (10).

§ 7. *Kinetic energy imparted to a solid body by convection.*—Suppose that the kinetic energy $\mathfrak{T}$ is imparted to the body by a motion of the invariable system XYZ. Suppose moreover that the equations of constraint, when expressed in terms of the relative coördinates of the body, do not involve the time explicitly. Then the velocities $(v_i)_1$ will satisfy the same conditions as the *actual relative velocities* of the body; in other words, it would be *possible* for this body to assume the velocity $(v_i)_1$ in its *relative* motion. On the other hand, it is clear that it would be *impossible* for the body to assume the velocity $(v_i)_2$ in its relative motion, on account of the condition that the equations of constraint

expressed in terms of the relative coördinates of the body do not involve the time explicitly. Hence the following theorem:

*The convective kinetic energy $\mathfrak{T}$ of a solid body, when the conditions of constraint expressed in terms of the relative coördinates do not involve the time explicitly, is the sum of the kinetic energy $\mathfrak{T}_1$ arising from a certain motion which would be possible relatively to the axes XYZ, and of the kinetic energy $\mathfrak{T}_2$ due to a certain other motion which it would be impossible for the body to assume relatively to these axes.**

§8. *Mechanical interpretation of the function G_2.*—I will now apply the results of the preceding paragraphs to the perturbative function of convective motion.

We have found in §6, chap. IV, that $\eta_1, \cdots, \eta_6$ being the six parameters defining the relative position of a solid body we shall have:

$$G = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \frac{\partial L}{\partial \eta_i'} \frac{\partial L}{\partial \eta_j'}. \tag{32}$$

If we take

$$v_i = \sum_{j=1}^{j=6} M_{ij} \frac{\partial L}{\partial \eta_j'}, \tag{33}$$

this expression of G will become:

$$G = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} v_i v_j, \tag{34}$$

and the formulas of §§2, 3 are immediately applicable. Thus, we shall have first

$$\mathfrak{V}^{(k)} = \frac{\partial L}{\partial q_k'}, \qquad \mathfrak{V}(\psi_k) = L(\psi_k),$$

and further,

$$G = G_1 + G_2, \tag{35}$$

$$\begin{aligned} G_1 &= \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (v_i)_1 (v_j)_1 = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \left(\frac{\partial L}{\partial \eta_i'}\right)_1 \left(\frac{\partial L}{\partial \eta_j'}\right)_1 \\ &= \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'}, \end{aligned} \tag{36}$$

$$\begin{aligned} G_2 &= \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (v_i)_2 (v_j)_2 = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \left(\frac{\partial L}{\partial \eta_i'}\right)_2 \left(\frac{\partial L}{\partial \eta_j'}\right)_2 \\ &= \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} L(\psi_k) L(\psi_e). \end{aligned} \tag{37}$$

* See the second footnote on p. 150 and *Example*, §13, chap. IX.

These formulas show that G_2 is nothing else than the third term of the perturbative function of convective motion, so designated before.

We have moreover arrived at the following mechanical interpretation * of the function G_2:

G_2 is the kinetic energy of the solid body due to constraint velocities arising from convective rotation.

§9. A slightly different form will now be given to the results of the preceding paragraph.

Let A be a point rigidly connected with the solid body. We know that the motion of the body may be decomposed into a translation of the point A and a rotation about this point. Let v_a be the velocity of A and let ω represent both the instantaneous axis and the angular velocity of rotation of the body. It is clear that we may apply to v_a and to ω the principle of decomposition into specific virtual and constraint velocities as we have applied it to the velocities v_i before. In this manner ω will be now the diagonal of a parallelogram constructed on $(\omega)_1$ and $(\omega)_2$, where $(\omega)_1$ represents both the specific virtual angular velocity and the specific virtual instantaneous axis of rotation of the body, while $(\omega)_2$ stands for both the angular constraint velocity and the instantaneous constraint axis of rotation of the body. The meaning of these terms is clear from the explanation and use of similar terms on previous occasions. Namely, $(\omega)_1$ would be obtained if the time, as far as it enters *explicitly* in the equations of constraint, became constant at the moment considered, while $(\omega)_2$ arises from the variation of the time as far as it enters *explicitly* in these equations.†

§10. In particular, let us now consider the convective rotation of the solid body. Constraint velocities arise from divers sources, as was pointed out in §9, chap. VIII. We are here concerned only with those arising from convective rotation.‡ Let therefore ω, $(\omega)_1$, $(\omega)_2$ represent respectively: the angular velocity and instantaneous axis of convective rotation, the specific virtual angular velocity and specific virtual instantaneous axis of convective rotation, the angular constraint velocity and instantaneous constraint axis of convective rotation. Then

$$G = I_\omega \omega^2, \quad G_1 = I_{\omega_1}(\omega)_1^2, \quad G_2 = I_{\omega_2}(\omega)_2^2, \tag{38}$$

where I_ω, I_{ω_1}, I_{ω_2} denote the moments of inertia of the body with regard to the instantaneous axes ω, $(\omega)_1$, $(\omega)_2$ respectively, and we shall have:

$$I_\omega \omega^2 = I_{\omega_1}(\omega)_1^2 + I_{\omega_2}(\omega)_2^2. \tag{39}$$

* See §9 of the preceding chapter.

† See NOTE at the close of the preceding chapter.

‡ Convective translation and the explicit appearance of the time in the equations of constraint expressed in terms of *relative* coördinates also give rise to constraint velocities.

It is clear that the three axes ω, $(\omega)_1$, $(\omega)_2$ lie in the same plane and pass through the same point; but *this point does not generally coincide with the origin of the axes* XYZ about which the convective rotation of the body takes place.

When the body can rotate freely about a point on the instantaneous axis ω, or about the origin of the axes XYZ, it is obvious that the angular velocity $(\omega)_2$ vanishes and $(\omega)_1$ coincides with ω. Therefore, in these cases $G = G_1$ and $G_2 = 0$. Thus we have found again the theorem of §3, chap. VII.

§11. If we assume that the equations of constraint expressed in the relative coördinates of the body do not involve the time explicitly, it is clear that a rotation about the axis $(\omega)_1$ would be *possible* in the relative motion of the body, while it would be *impossible* for this body to rotate about the axis $(\omega)_2$ in its relative motion. Hence, we may say that

When the conditions of constraint expressed in the relative coördinates of the body do not involve the time explicitly, then G_2 is the kinetic energy of the body due to convective rotation about an axis about which relative rotation would be impossible.

§12. Let E stand for the ellipsoid of inertia of the body at the point of intersection of the three axes ω, $(\omega)_1$, $(\omega)_2$. It is obvious from (39) that *the axis* $(\omega)_2$ *lies in a plane conjugate to the direction* $(\omega)_1$ with regard to the ellipsoid E, and similarly that *the axis* $(\omega)_1$ *lies in a plane conjugate to the direction* $(\omega)_2$ with regard to the same ellipsoid.

If the direction of either $(\omega)_1$ or $(\omega)_2$ be that of an axis of the ellipsoid E, the instantaneous axes $(\omega)_1$ and $(\omega)_2$ are at right angles to each other.

In this connection the following remark may be in place. Let o be the origin of a system of axes a_1, a_2, a_3 whose directions form *a conjugate system* with respect to the ellipsoid of inertia of the body at the point o. If Ω denote the angular velocity of rotation of the body in magnitude and direction, and Ω_1, Ω_2, Ω_3 the components of Ω along the axes a_1, a_2, a_3; if moreover I_Ω, I_{a_1}, I_{a_2}, I_{a_3} be the moments of inertia of the body about the axes Ω, a_1, a_2, a_3; then the kinetic energy of rotation of the body about the point o will be

$$\tfrac{1}{2} I_\Omega \Omega^2 = \tfrac{1}{2}(I_{a_1}\Omega_1^2 + I_{a_2}\Omega_2^2 + I_{a_3}\Omega_3^2). \tag{40}$$

When the directions a_1, a_2, a_3 are those of the *axes* of the ellipsoid of inertia, and λ_1, λ_2, λ_3 denote the cosines of the angles which Ω forms with these axes, the expression (40) assumes the well-known form:

$$\tfrac{1}{2} I_\Omega \Omega^2 = \tfrac{1}{2}(I_{a_1}\lambda_1^2 + I_{a_2}\lambda_2^2 + I_{a_3}\lambda_3^2)\Omega^2. \tag{41}$$

Applying this remark to the case under discussion, when the convective rotation of the body alone is considered, so that $\frac{1}{2} I_\Omega \Omega^2 = G$, we arrive at the following conclusions:

If the body in its relative motion rotates about two axes a_2 and a_3 whose directions are conjugate with respect to the ellipsoid E, then G_2 is the kinetic energy of convective rotation of the body about the axis a_1 conjugate to a_2 and a_3. Denoting by ω' the component of ω along the axes a_1 we shall have

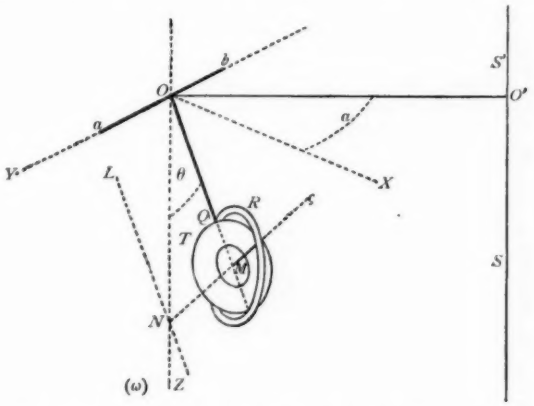
$$G_2 = \tfrac{1}{2} I_{a_1} \omega'^2.$$

If the body in its relative motion is prevented from rotating about two axes a_2 and a_3 whose directions are conjugate with respect to the ellipsoid E, then G_1 is the kinetic energy of convective rotation of the body about the axis a_1 conjugate to a_2 and a_3 and we shall have

$$G_1 = \tfrac{1}{2} I_{a_1} \omega'^2.$$

As an illustration let us apply these results to an example.

§13. *Example. Sire's gyroscopic pendulum.*—The pendulum consists of a shaft OQ, a ring R, and a tore T, the latter revolving about a diameter of the ring. The pendulum swings about a horizontal axis ab which forms a fixed



angle $\frac{1}{2}\pi - a$ with OO', a horizontal line of fixed length. The whole apparatus revolves about the vertical axis SS' with the angular velocity ω. The axes XYZ and their origin O will be selected as on the adjoining figure. The three axes $M\zeta$ (of the tore), OM (of the shaft) and OY (or ab) are a tright angles to each other. Denote by C and A respectively the axial and equatorial moments of inertia of the tore, by θ the angle between the axis OM and the

vertical OZ, and by N the point of intersection of the axes $M\zeta$ and OZ. Draw NL parallel to the axis OM.

The tore rotates about two axes, one being its physical axis and the other perpendicular to the plane ZOM. Hence $(\omega)_1$ lies in the plane which passes through $M\zeta$. We thus obtain the point N as the intersection of ω and $(\omega)_1$. Now, the directions of $N\zeta$ and of the line perpendicular to the plane ZOM being the directions of two axes of the ellipsoid of inertia of the tore at N, and these axes being at the same time those about which the relative rotation of the tore takes place, *G_2 will be the kinetic energy of convective rotation of the body about the third axis of the ellipsoid, namely the line NL.*

The moment of inertia of the body about this line being $(md^2tg^2\theta + A)$, where m is the mass of the tore and $d = OM$, and, on the other hand, the component of ω along the line LN being $\omega\cos\theta$, we shall have:

$$G_2 = \tfrac{1}{2}(md^2\sin^2\theta + A\cos^2\theta)\omega^2.$$

CHAPTER X.

Other Methods of obtaining the Differential Equations of Relative Motion.

§1. *Differential equations of absolute motion.*—The position of a material system with regard to the axes ΞHZ, fixed in space, will be given when its position relatively to the moving axes XYZ and the motion of the latter relatively to the axes ΞHZ are given. Hence, assuming that the motion of the axes XYZ is known, the system of coördinate parameters may be considered as defining the *absolute* as well as the *relative* motion of the material system. From this point of view relative motion and the differential equations defining it constitute in no way a theory of their own. In fact, to define the relative motion we may simply make use of the differential equations of absolute motion. Thus, if $T^{(a)}$ denote the kinetic energy of the material system due to its absolute motion, we shall have the differential equations of Lagrange:

$$(1) \qquad p_k^{(a)} = \frac{\partial T^{(a)}}{\partial q_k'}, \qquad \frac{dp_k^{(a)}}{dt} = \frac{\partial(T^{(a)} + U)}{\partial q_k} \qquad (k = 1, 2, \cdots, \mu),$$

and those of Hamilton-Jacobi:

$$(2) \qquad \begin{cases} \dfrac{dp_k^{(a)}}{dt} = -\dfrac{\partial H^{(a)}}{\partial q_k}, \qquad \dfrac{dq_k}{dt} = \dfrac{\partial H^{(a)}}{\partial p_k^{(a)}} \\ H^{(a)} = \sum_{k=1}^{k=\mu} p_k^{(a)} q_k' - T^{(a)} - U \end{cases} \qquad (k = 1, 2, \cdots, \mu),$$

which, on integration, give the $q_1, \cdots, q_\mu$ as functions of t and 2μ arbitrary constants, thus solving the problem of relative motion.

§2. *Lagrange's equations for relative motion derived from the equations of absolute motion.*—It will now be shown how the differential equations of chaps. I, II may be obtained from the equations (1) and (2). I will introduce two new functions, namely,

$$N = \sum m_i(v_{0x}x_i + v_{0y}y_i + v_{0z}z_i) = Mv_0r_c \cos(v_0, r_c), \tag{3}$$

$$T_0 = \tfrac{1}{2} Mv_0^2, \tag{4}$$

where v_0 denotes the velocity of the origin of the axes XYZ, while r_c, M as before denote the radius vector of the center of inertia and the total mass of the material system respectively. It can be easily verified that

$$T^{(a)} = T^{(r)} + K + L + G + T_0 + \frac{dN}{dt}. \tag{5}$$

Now, since the functions K, G and T_0 do not involve the derivatives $q_1', \cdots, q_\mu'$, and moreover since

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \sum_{k=1}^{k=\mu} q_k' \frac{\partial N}{\partial q_k}, \tag{6}$$

and therefore

$$\frac{\partial}{\partial q_k'}\left(\frac{dN}{dt}\right) = \frac{\partial N}{\partial q_k}, \tag{7}$$

it will be seen that

$$p_k^{(a)} = \frac{\partial T^{(a)}}{\partial q_k'} = \frac{\partial}{\partial q_k'}\left(T^{(r)} + L + \frac{dN}{dt}\right) = \mathfrak{p}_k + \frac{\partial N}{\partial q_k'}. \tag{8}$$

Hence,

$$\frac{d\mathfrak{p}_k}{dt} = \frac{\partial(T^{(a)} + U)}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial N}{\partial q_k}\right).$$

But it can be readily shown that

$$\frac{d}{dt}\left(\frac{\partial N}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{dN}{dt}\right), \tag{9}$$

and therefore the preceding equation takes the form:

$$\frac{d\mathfrak{p}_k}{dt} = \frac{\partial}{\partial q_k}\left(T^{(a)} + U - \frac{dN}{dt}\right) = \frac{\partial(T^{(r)} + L + U_1)}{\partial q_k},$$

the function T_0 not involving the $q_1, \cdots, q_\mu$. The function U_1 has the same meaning here as in chap. I.

Thus equations (10′) of chap. I have been deduced from equations (1) of this chapter.

Proceeding in like manner when the real forces have no potential function, equations (10), chap. I may be obtained from the following equations of absolute motion:

$$(10) \qquad p_k^{(a)} = \frac{\partial T^{(a)}}{\partial q_k'}, \quad \frac{dp_k^{(a)}}{dt} = \frac{\partial T^{(a)}}{\partial q_k} + Q_k^{(0)} \qquad (k = 1, 2, \cdots, \mu).$$

§ 3. *Hamilton-Jacobi's equations derived from the equations of absolute motion.*—Passing to the canonical equations (2) I remark that

$$H^{(a)} = \sum_{k=1}^{k=\mu} p_k q_k' + \sum_{k=1}^{k=\mu} \frac{\partial L}{\partial q_k'} q_k' + \sum_{k=1}^{k=\mu} \frac{\partial N}{\partial q_k} q_k'$$
$$- T^{(r)} - K - L - G - T_0 - \frac{dN}{dt} - U,$$

or, since*

$$\sum_{k=1}^{k=\mu} \frac{\partial L}{\partial q_k'} q_k' = L - \lambda, \qquad \sum_{k=1}^{k=\mu} \frac{\partial N}{\partial q_k} q_k' = \frac{dN}{dt} - \frac{\partial N}{\partial t},$$

that

$$H^{(a)} = \sum_{k=1}^{k=\mu} p_k q_k' - T^{(r)} - U_1 - \lambda - \frac{\partial N}{\partial t} - T_0.$$

Introducing into this expression the variables $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$ in lieu of the $p_1, \cdots, p_\mu$, it will become:

$$(11) \qquad H^{(a)} = \Theta_1 - U_1 - \lambda - \frac{\partial N}{\partial t} - T_0,$$

Θ_1 having the same meaning as in chap. II.

Now, in the equations (2), $H^{(a)}$ is a function of the $p_1^{(a)}, \cdots, p_\mu^{(a)}$, whereas in (11) it is a function of the $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$. Denoting by the symbol $[\ ]_p$ that the function inside the brackets is expressed in terms of the $p_1, p_2, \cdots$, we shall have:

$$\frac{\partial [H^{(a)}]_{p^{(a)}}}{\partial q_k} = \frac{\partial [H^{(a)}]_p}{\partial q_k} - \sum_{e=1}^{e=\mu} \frac{\partial [H^{(a)}]_{p^{(a)}}}{\partial p_e^{(a)}} \frac{\partial}{\partial q_k} \left(\frac{\partial N}{\partial q_e} \right)$$
$$= \frac{\partial [H^{(a)}]_p}{\partial q_k} - \sum_{e=1}^{e=\mu} q_e' \frac{\partial}{\partial q_k} \left(\frac{\partial N}{\partial q_e} \right)$$
$$= \frac{\partial [H^{(a)}]_p}{\partial q_k} - \frac{\partial}{\partial q_k} \sum_{e=1}^{e=\mu} q_e' \frac{\partial N}{\partial q_e} = \frac{\partial (\Theta_1 - U_1 - \lambda)}{\partial q_k} - \frac{\partial}{\partial q_k} \left(\frac{dN}{dt} \right),$$

and therefore, by (2),

* See formulas (2), chap. II and (8) of this chapter.

$$\frac{dp_k^{(a)}}{dt} = -\frac{\partial(\Theta_1 - U_1 - \lambda)}{\partial q_k} + \frac{\partial}{\partial q_k}\left(\frac{dN}{dt}\right).$$

But

$$\frac{dp_k^{(a)}}{dt} = \frac{d\mathfrak{p}_k}{dt} + \frac{d}{dt}\left(\frac{\partial N}{\partial q_k}\right) = \frac{\partial \mathfrak{p}_k}{dt} + \frac{\partial}{\partial q_k}\left(\frac{dN}{dt}\right).$$

Hence

$$\frac{d\mathfrak{p}_k}{dt} = -\frac{\partial(\Theta_1 - U_1 - \lambda)}{\partial q_k} \qquad (k=1, 2, \cdots, \mu).$$

At the same time, as is readily seen from (2) and (11),

$$\frac{dq_k}{dt} = \frac{\partial(\Theta_1 - U_1 - \lambda)}{\partial \mathfrak{p}_k} \qquad (k=1, 2, \cdots, \mu).$$

These 2μ equations are identical with the canonical system (6), chap. II.

When the real forces have no potential function, the system of differential equations (4), (5) may be obtained in like manner from the following equations of absolute motion:

$$(12) \qquad \begin{cases} \dfrac{dp_k^{(a)}}{dt} = -\dfrac{\partial\Theta^{(a)}}{\partial q_k} + Q_k^{(0)}, \qquad \dfrac{dq_k}{dt} = \dfrac{\partial\Theta^{(a)}}{\partial p_k^{(a)}} \quad (k=1, 2, \cdots, \mu). \\ \Theta^{(a)} = \sum_{k=1}^{k=\mu} p^{(a)} q_k' - T^{(a)}. \end{cases}$$

§4. *Gilbert's method.*—PH. GILBERT * has indicated an exceedingly simple way of obtaining LAGRANGE'S equations of relative motion. It is based on the remark that *by adding the function K to the potential U we may assume that the origin of the axes XYZ is fixed in space.* Then $T^{(a)}$ becomes what has been denoted by T_2 in chap. I, and the differential equations of (absolute) motion will be:

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial q_k'}\right) = \frac{\partial(T_2 + U + K)}{\partial q_k} \qquad (k=1, 2, \cdots, \mu).$$

which are equivalent to the equations (10′), chap. I, as was there explained.

§5. While GILBERT restricted himself to LAGRANGE'S equations it is easy to see how his method may be applied to the canonical system of HAMILTON-JACOBI. In fact, if we put

$$H_2 = \sum_{k=1}^{k=\mu} \mathfrak{p}_k q_k' - T_2 - (U + K),$$

where, again,

$$\mathfrak{p}_k = \frac{\partial T_2}{\partial q_k'},$$

* *Mémoire sur l'Application de la méthode de Lagrange*, etc. Part I, §1.

we shall arrive at the canonical system of differential equations:

$$\frac{d\mathfrak{p}_k}{dt} = -\frac{\partial H_2}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial H_2}{\partial \mathfrak{p}_k} \qquad (k=1, 2, \cdots, \mu); \tag{13}$$

and it is readily seen that

$$H_2 = \Theta_1 - U_1 - \lambda. \tag{14}$$

To show this, I remark that

$$\begin{aligned} H_2 &= \sum_{k=1}^{k=\mu} p_k q_k' + \sum_{k=1}^{k=\mu} \frac{\partial L}{\partial q_k'} q_k' - T_2 - U - K \\ &= \sum_{k=1}^{k=\mu} p_k q_k' + L - \lambda - T_2 - U - K \\ &= \sum_{k=1}^{k=\mu} p_k q_k' - T^{(r)} - U_1 - \lambda. \end{aligned}$$

Introducing the variables $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$ into this expression, $\sum p_k q_k' - T^{(r)}$ will become what has been denoted by Θ_1, and H_2 will assume the form (14).

Equations (13), (14) are identical with equations (6), chap. II.

CHAPTER XI.

Some remarks concerning the Differential Equations of Lagrange and of Hamilton-Jacobi.

§1. We have seen in chapter I that the fictitious forces of Coriolis give rise to two classes of terms in the expression of Q_k. One class can be expressed as the derivatives with regard to the q_k of a function $(K+G)$ of the variables $q_1, q_2, \cdots, q_\mu, t$; while the other is capable of being thrown into the form:

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_k'}\right),$$

L being a function of $q_1, q_2, \cdots, q_\mu; q_1', q_2', \cdots, q_\mu'; t$. That part of Q_k, therefore, which is due to the fictitious forces of Coriolis may be omitted if, in exchange we add the function $K+G$ to the potential of the real forces and at the same time add the function L to the kinetic energy $T^{(r)}$ of relative motion. This in fact is the way in which we derived equations (9) and (10) from (1) in chap. I.

In general, let the equations:

$$\frac{d}{dt}\frac{\partial T}{\partial q_k'} - \frac{\partial T}{\partial q_k} = Q_k \qquad (k=1, 2, \cdots, \mu) \tag{1}$$

define the motion (absolute or relative) of a material system. It is clear that any part of Q_k such that it may be thrown into the form:

$$\frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial q_k'},$$

V being a function of $q_1, \cdots, q_\mu; q_1', \cdots, q_\mu'; t$, may be omitted if in exchange we add the function V to the function T in (1). We thus arrive at the extended notion of a force function as introduced into mechanics by SCHERING.* Namely, if the forces F_i are such that

$$(2) \quad \sum_{i=1}^{i=n}\left(F_{ix}\frac{\partial x_i}{\partial q_k} + F_{iy}\frac{\partial y_i}{\partial q_k} + F_{iz}\frac{\partial z_i}{\partial q_k}\right) = \frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial q_k'} \qquad (k=1, 2, \cdots, \mu),$$

V being a function of $q_1, \cdots, q_\mu; q_1', \cdots, q_\mu'; t$, then LAGRANGE'S equations will take the form:

$$p_k = \frac{\partial(T+V)}{\partial q_k'}, \qquad \frac{dp_k}{dt} = \frac{\partial(T+V)}{\partial q_k} \qquad (k=1, 2, \cdots, \mu).$$

§2. From the above it follows that

If the function V satisfies the conditions:

$$(3) \quad \frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial q_k'} = 0, \qquad (k=1, 2, \cdots, \mu),$$

then we may add or subtract this function from T without in any way affecting the motion of a material system given by the differential equations:

$$\frac{d}{dt}\frac{\partial T}{\partial q_k'} - \frac{\partial T}{\partial q_k} = Q_k.$$

In fact such an operation is equivalent to the introduction of vanishing forces.

§3. As seen from formulas (7) and (9) of the preceding chapter, the function dN/dt satisfies the conditions (3). We may therefore, without affecting in any way the motion of the given system, add dN/dt to $T^{(r)}$ in the equations (10) or (10′) of chap. I, or subtract dN/dt from $T^{(a)}$ in the equations (1) of chap. X. The first of these operations gives:

---

* *Abhandlungen der Kgl. Ges. d. Wissenschaften zu Göttingen*, vol. 18, §§ 16-18.

The force function of SCHERING is defined by the equation:

$$(2') \quad \sum F_i \delta r_i = \delta V - \frac{d}{dt}\sum_{i=1}^{i=3n}\frac{\partial V}{\partial u_i}\delta u_i,$$

where u_i has the same meaning as in this paper, while r_i stands for $\sqrt{x_i^2 + y_i^2 + z_i^2}$. The μ equations (2) are equivalent to equation (2′).

It is by means of SCHERING'S extended notion of a force function that KAMERLINGH-ONNES derived the equations (10′) of chap. I and (6) of chap. II.

$$\frac{d}{dt}\frac{\partial}{\partial q_k'}\left(T^{(r)} + L_1' + \frac{dN}{dt}\right) = \frac{\partial}{\partial q_k}\left(T^{(r)} + K + L + G + \frac{dN}{dt}\right) + Q_k^0,$$

or, as is obvious,

$$\frac{dp_k^{(a)}}{dt} = \frac{\partial T^{(a)}}{\partial q_k} + Q_k^0. \tag{4}$$

If the real forces have a potential function U,

$$\frac{dp_k^{(a)}}{dt} = \frac{\partial(T^{(a)} + U)}{\partial q_k}. \tag{5}$$

These are exactly the equations (1) of chap. X. The second operation, as may readily be seen, will change equations (4) or (5) of this chapter into equations (10) and (10′) of chap. I.

Thus, the remark of §2 enables us to pass at once from the equations of absolute motion to those of relative motion and vice versa.

§4. In a similar way can be brought forth the connection between the canonical systems (6) of chap. II and (2) of chap. X.

In fact, in equations (2) of chap. X we may subtract dN/dt from $T^{(a)}$ without in any way affecting the motion. Thus we shall have:

$$\frac{d\mathfrak{p}_k}{dt} = -\frac{\partial H'}{\partial q_k}, \qquad \frac{dq_k}{dt} = \frac{\partial H'}{\partial \mathfrak{p}_k}, \tag{6}$$

because now

$$p_k^{(a)} = \frac{\partial}{\partial q_k'}\left(T^{(a)} - \frac{dN}{dt}\right) = \mathfrak{p}_k;$$

and as to H', it will be

$$\begin{aligned} H' &= \sum_k q_k' \frac{\partial}{\partial q_k'}\left(T^{(a)} - \frac{dN}{dt}\right) - T^{(a)} + \frac{dN}{dt} - U \\ &= \sum_k \mathfrak{p}_k q_k' - T^{(r)} - L - T_0 - U_1 \\ &= \sum_k p_k q_k' - T^{(r)} - T_0 - U_1 - \lambda \\ &= \Theta_1 - U_1 - \lambda. \end{aligned}$$

Here we may neglect the term T_0, as derivatives of H' with respect to q_k and $\mathfrak{p}_k$ only enter into (6) and H' is expressed as a function of $q_1, \cdots, q_\mu; \mathfrak{p}_1, \cdots, \mathfrak{p}_\mu; t$.

Hence, as anticipated, equations (6) are identical with the system (6) of chap. II.

Conversely, by adding dN/dt to $T^{(r)}$ we can obtain from equations (4), (5), (6) of chap. II the system (2) of chap. X, or a corresponding system for the case when the real forces do not have a potential function.

§5. If, considering again, more generally, the case of any motion, as expressed by the equations (1) of the present chapter, we apply the remark made in §1 to the canonical system:

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial H}{dp_k} \qquad (k = 1, 2, \cdots, \mu),$$

$$H = \sum_k p_k q_k' - T - U,$$

we shall obtain for the generalized force function of SCHERING the canonical system of differential equations in the following form:*

$$p_k = \frac{\partial (T + V)}{\partial q_k'}, \quad H = \sum_{k=1}^{k=\mu} p_k q_k' - T - V;$$

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \qquad (k = 1, 2, \cdots, \mu).$$

§6. Throughout this paper, whenever Lagrangian coördinates were introduced, it was assumed that they formed an *independent* system, i. e., the number μ of such coördinates was equal to the degree of freedom of the material system. It is needless to say that to define the relative position of this system a number $\mu' > \mu$ of coördinates could be chosen, but then certain Lagrangian factors due to the reactions of some or all of the constraints would have to be introduced in the differential equations of motion. Putting $\mu' = \mu + \nu$ it would only be necessary to add the functions:

$$\sum_{k=1}^{k=\nu} \lambda_k \frac{\partial \phi_k}{\partial q_1}, \ \cdots; \quad \sum_{k=1}^{k=\nu} \lambda_k \frac{\partial \phi_k}{\partial q_e}, \ \cdots; \quad \sum_{k=1}^{k=\nu} \lambda_k \frac{\partial \phi_k}{\partial q_{\mu'}};$$

to $Q_1^{(0)}, \cdots, Q_e^{(0)}, \cdots, Q_{\mu'}^{(0)}$ respectively in the equations (1) chap. I and (1) chap. II.

A similar change would be necessary in the equations of motion of a solid body.

* SCHERING, loc. cit.

PLANE CUBICS AND IRRATIONAL COVARIANT CUBICS*

BY

HENRY S. WHITE

Introduction.

Every covariant containing two contragredient sets of variables, of the same degree in both sets of variables, may be interpreted as the symbol of a transformation. For the lineo-linear form indeed this is the most common interpretation, a collineation; and the symbol of such a transformation is applied to a form or quantic of order n as many times *simultaneously* as there are units in its order, n. Thus if the transformation be denoted by the lineo-linear form $a_x u_a$, and the quantic of the third order by A_x^3, the transformed cubic is denoted by

$$A_a A_{a'} A_{a''} a_x a'_x a''_x \equiv A_x'^3 .$$

Algebraically, however, we might describe this as the *complete* transformation; and no less interest would attach to the *partial* transformations, whose effect would be denoted by

$$A_a A_x^2 a_x = A_x'^3, \qquad A'_a A_x'^2 a_x = A_x''^3, \text{ etc.}$$

Similarly those mixed covariants of a ternary cubic which have their class and order equal may be applied: first, to effect complete transformation upon quantics of their own order or class; second, to effect partial transformation upon quantics of an order or class higher than their own. Employed in this way, they lead to the study of curves which are unchanged by such a transformation. These curves are properly described as curves irrationally covariant to the fundamental cubic.

In the theory of the ternary cubic there are seven mixed covariants of the specified sort, three of order and class (2, 2), and four of order and class (3, 3). Conics invariant under complete transformation of two of the three kinds (2, 2) have been discussed in a paper published in the preceding number of the Transactions. In the present paper it is proposed to consider cubics invariant under *partial* transformation by covariants (2, 2), and those invariant

* Presented to the Society (Chicago) December 29, 1899. Received for publication February 15, 1900.

under *complete* transformation by covariants (3, 3). There will remain for subsequent treatment the two sets of conics invariant under the third transformation (2, 2), and invariant curves of order higher than the third. Both these topics present points of interest and novelty.

The new covariant cubics found by this investigation are eight in number, all of the type called equianharmonics. Dually of course there are found eight corresponding curves of class three. In conclusion are discussed the relations of apolarity in these two sets, and the relation of each set to the four inflexional triangles.

§1. *Cubics connected with Covariant Conics.*

The ternary cubic has three linearly independent covariants of order 2 and class 2. These are denoted in CLEBSCH'S and GORDAN'S writings by Θ, H, and K. If we write a_x^3 for the fundamental cubic and a_x^3 for its Hessian, then these three have the symbolic structure:

$$\Theta = (aa'u)^2 a_x a'_x = \Theta_x^2 u_\vartheta^2 ,$$

$$\mathrm{H} = (aau)^2 a_x a_x = \mathrm{H}_x^2 u_\eta^2 ,$$

$$\mathrm{K} = (aa'u)^2 a_x a'_x = \mathrm{K}_x^2 u_\kappa^2 .$$

In the former paper I have employed Θ as a symbol of operation, and have discussed the conics that are transformed into themselves by that operation. Considering it first as a transformer of point loci, second as a transformer of line-loci, we found in the one case two nets, in the other two webs of conics which were termed respectively autopolo-conics and conic autopolars. These were the polar systems of two pairs of cubics, termed autopolo-cubics and cubic autopolars. Are there also other autopolo-cubics, and how are they defined geometrically? Neglecting, temporarily, the dual aspect of the transformers, we extend the inquiry also to point-loci of third order invariant to transformation by H and K respectively. Of these, two in each case have for polar-systems nets of conics invariant with respect to the same transformation. While these have peculiarly interesting features, we shall postpone their discussion and limit our inquiry to cubics whose conic polars are not invariant with respect to any rational covariant transformation of order and class lower than 3; such may be termed *special* covariant cubics.

§2. *Special Covariant Cubics whose Transformers are Reducible.*

As transformers of point loci consider three reducible forms $\Theta_x^2 u_\vartheta^2 u_x$, $\mathrm{H}_x^2 u_\eta^2 u_x$, and $\mathrm{K}_x^2 u_\kappa^2 u_x$. Call them Θ, H, K, as before, remembering that they are now

extended so as to be of order and class 3. Define them for use as transformers thus, the ternary cubic L_x^3 being indeterminate:

$$\Theta(L_x^3) = \Theta_x^2 L_\vartheta^2 L_x,$$
$$\mathrm{H}(L_x^3) = \mathrm{H}_x^2 L_\eta^2 L_x,$$
$$\mathrm{K}(L_x^3) = \mathrm{K}_x^2 L_\kappa^2 L_x.$$

We are to determine cubics L_x^3 satisfying the condition of invariantiveness with respect to each of these transformations in turn.

First: the operator Θ.

We impose upon L_x^3 the condition:

$$\Theta(L_x^3) = \Theta_x^2 L_\vartheta^2 L_x \equiv \lambda \cdot L_x^3. \tag{1}$$

The fundamental cubic being taken in the canonical form of HESSE:

$$a_x^3 = x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3, \tag{2}$$

this identity is equivalent to the following ten equations:

$$\left.\begin{aligned} 2m^2L_{111} + 4mL_{123} &= \lambda \cdot L_{111},\\ 2m^2L_{222} + 4mL_{123} &= \lambda \cdot L_{222},\\ 2m^2L_{333} + 4mL_{123} &= \lambda \cdot L_{333},\\ 2(L_{111} + L_{222} + L_{333}) + 12m^2L_{123} &= -6\lambda \cdot L_{123}; \end{aligned}\right\} \text{(I)}$$

$$\left.\begin{aligned} -2m^2L_{112} + 4mL_{223} - 2L_{133} &= 3\lambda \cdot L_{112},\\ -2m^2L_{223} + 4mL_{133} - 2L_{112} &= 3\lambda \cdot L_{223},\\ -2m^2L_{133} + 4mL_{112} - 2L_{223} &= 3\lambda \cdot L_{133}; \end{aligned}\right\} \text{(II)}$$

$$\left.\begin{aligned} -2m^2L_{122} + 4mL_{113} - 2L_{233} &= 3\lambda \cdot L_{122},\\ -2m^2L_{233} + 4mL_{122} - 2L_{113} &= 3\lambda \cdot L_{233},\\ -2m^2L_{113} + 4mL_{233} - 2L_{122} &= 3\lambda \cdot L_{113}. \end{aligned}\right\} \text{(III)}$$

We seek a solution $(L_{111}, \cdots, L_{333}, \lambda)$ in which not all the coefficients L_{ijk} are 0. The ten coefficients enter, four in the equations I, three in the equations II, and three in the equations III. Unless all its coefficients L are 0 each group of equations determines values of λ. The groups II and III evidently determine in this way the same three values of λ; these values are distinct from those determined by group I.

Ad I) Unless $L_{123} = 0$, it appears from inspection of equations (I) that

$$L_{111} = L_{222} = L_{333}, \tag{3}$$

and that λ is a root of the quadratic equation:

$$\lambda^2 = 4(m^4 - m) = \tfrac{1}{6} S, \tag{4}$$

S denoting the well-known quartic invariant of the fundamental cubic. To find the equation $L_x^3 = 0$ corresponding to these two discrete values of λ, let

$$\frac{L_{123}}{L_{111}} = \frac{L_{123}}{L_{222}} = \frac{L_{123}}{L_{333}} = L,$$

and eliminate λ from the fourth and any other of equations (I). The result:

$$4mL(m + L) + 1 = 0, \tag{5}$$

is the equation found in the former paper* characterizing the two cubics (A and B) which have the same Cayleyan curve with the fundamental cubic, and whose conic polars all are invariant to the transformer Θ.

For *special* autopolo-cubics, therefore, we must examine the remaining (eight) values of λ.

If $L_{123} = 0$, we have from group (I):

$$\lambda = 2m^2, \qquad L_{111} + L_{222} + L_{333} = 0: \tag{6, 7}$$

and the corresponding sheaf of cubics has the equation:

$$(c_2 - c_3)x_1^3 + (c_3 - c_1)x_2^3 + (c_1 - c_2)x_3^3 = 0, \tag{8}$$

where c_1, c_2, and c_3 are arbitrary parameters. The discussion of this sheaf of cubics may be postponed until the equations (II) and (III) have been examined.

Ad II) From the cyclic character of group (II) it is seen that the following sets of values will satisfy these conditions:

$$L_{112} = L_{223} = L_{331}, \qquad \lambda = -\tfrac{2}{3}\cdot(m-1)^2; \tag{IIa}$$

$$L_{112} = \omega^2 L_{223} = \omega L_{331}, \qquad \lambda = -\tfrac{2}{3}\cdot(m-\omega)^2; \tag{IIb}$$

$$L_{112} = \omega L_{223} = \omega^2 L_{331}, \qquad \lambda = -\tfrac{2}{3}\cdot(m-\omega^2)^2. \tag{IIc}$$

Here ω and ω^2 denote the imaginary cube roots of unity.

Ad III) Similarly the solutions of group (III) are found to be the following:

* loc. cit., p. 3, formulas (7) and (9).

$$(\text{III}a) \qquad L_{122} = L_{233} = L_{311}, \qquad \lambda = -\tfrac{2}{3}\cdot(m-1)^2;$$

$$(\text{III}b) \qquad L_{122} = \omega^2 L_{233} = \omega L_{311}, \qquad \lambda = -\tfrac{2}{3}\cdot(m-\omega)^2;$$

$$(\text{III}c) \qquad L_{122} = \omega L_{233} = \omega^2 L_{311}, \qquad \lambda = -\tfrac{2}{3}\cdot(m-\omega^2)^2.$$

Accordingly the roots λ of (II) and (III) are equal in pairs, and there is one pair of equal values λ in group (I). These pairs of equal roots indicate four singly infinite sheaves of autopolo-cubics, namely the following:

$$(9a) \qquad c_1(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) + c_2(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) = 0, \qquad [\lambda = -\tfrac{2}{3}(m-1)^2];$$

$$(9b) \qquad c_1(x_1^2x_2 + \omega x_2^2x_3 + \omega^2x_3^2x_1) + c_2(x_1x_2^2 + \omega x_2x_3^2 + \omega^2x_3x_1^2) = 0, \qquad [\lambda = -\tfrac{2}{3}(m-\omega)^2];$$

$$(9c) \qquad c_1(x_1^2x_2 + \omega^2x_2^2x_3 + \omega x_3^2x_1) + c_2(x_1x_2^2 + \omega^2x_2x_3^2 + \omega x_3x_1^2) = 0, \qquad [\lambda = -\tfrac{2}{3}(m-\omega^2)^2];$$

$$(9d) \qquad (c_2 - c_3)x_1^3 + (c_3 - c_1)x_2^3 + (c_1 - c_2)x_3^3 = 0, \qquad [\lambda = 2m^2].$$

The autopolo-cubics of a non-singular plane cubic are the two discrete cubics whose conic polars are autopolo-conics of the fundamental curve, and four sheaves of special covariant cubics, all equianharmonic, whose equations for the canonical form are written as (9*a*, 9*b*, 9*c*, 4*d*) *above.*

Second: the operator K.

The discussion of the operator Θ may be repeated for the operator K with no other change than the substitution of $-(1 + 2m^3)/6m^2$, the parameter of the Hessian, in place of m, the parameter of the fundamental cubic. The values of λ compatible with the condition:

$$K_x^2 L_\kappa^2 L_x = \lambda \cdot L_x^3,$$

are accordingly the two discrete roots of the quadratic equation:

$$(10) \qquad \lambda^2 = \tfrac{1}{12}(8T^2 - S^3),$$

and the following four, each counted doubly:

$$(11) \qquad \begin{aligned} \lambda &= -\tfrac{2}{3}(1 - 6m^2 + 2m^3)^2, \\ \lambda &= -\tfrac{2}{3}(1 - 6\omega m^2 + 2m^3)^2, \\ \lambda &= -\tfrac{2}{3}(1 - 6\omega^2 m^2 + 2m^3)^2, \\ \lambda &= 2(1 + 2m^3)^2. \end{aligned}$$

The two discrete roots will give two cubics of the syzygetic sheaf, non-special, different from those found by the operator Θ; but the four sheaves of cubics

corresponding to the remaining roots are the same for both operators, since their equations ($9a$, $9b$, $9c$, $9d$) are independent of the parameter m; while the fundamental cubic and its Hessian are reduced to Hesse's canonical form simultaneously and K is the operator Θ formed from the Hessian instead of the fundamental cubic.

The cubics invariant with respect to the operator K *are the autopolocubics of the Hessian, and consist of two discrete curves whose conic polars are the autopolo-conics of the Hessian, together with the same four simply infinite sheaves of cubics that are invariant with respect to the operator* Θ.

Third: the operator H.

Expanding for the canonical form of the cubic the required identity:

$$\mathrm{H}_x^2 L_\eta^2 L_x \equiv \lambda \cdot L_x^3, \tag{12}$$

we have ten equations closely resembling groups (I), (II), (III) under the operator Θ. From four typical examples the entire set may be obtained by permutation of indices:

(IV)
$$2m\mu L_{111} + 2(m+\mu)\, L_{123} = \lambda \cdot L_{111}, \qquad \text{(3 equations)};$$
$$-2(L_{111} + L_{222} + L_{333}) - 12m\mu L_{123} = 6\lambda L_{123}, \qquad \text{(1 equation)};$$

(V)
$$-2m\mu L_{112} + 2(m+\mu)L_{223} - 2L_{133} = 3\lambda L_{112}, \qquad \text{(3 equations)};$$

(VI)
$$-2m\mu L_{113} + 2(m+\mu)L_{233} - 2L_{122} = 3\lambda L_{113}, \qquad \text{(3 equations)};$$

μ denoting the parameter of the Hessian.

Ad IV) Equating to zero as before all coefficients L_{ijk} which do not occur in those equations, for otherwise none of the others yield the same values of λ, we find two solutions which are not special, and two equal roots λ indicating a sheaf of special cubics. The former are found by assuming

$$L_{123} \neq 0,$$

whence

$$\lambda = \pm 4m(1 - m^3), \tag{13}$$
$$L_{111} = L_{222} = L_{333};$$

then if we set

$$\frac{L_{123}}{L_{111}} = L,$$

we have the relation:

$$1 + 4m\mu \cdot L + 2(m+\mu) \cdot L^2 = 0. \tag{14}$$

This relation is exactly that which gives the parameters of two nets of conics

invariant with respect to the transformation H, the polar nets of the cubics here found. These therefore are not *special* covariant cubics.

Special cubics are found by taking

$$L_{123} = 0,$$

whence

$$L_{111} + L_{222} + L_{333} = 0,$$

$$\lambda = 2m\mu = -2m(1 + 2m^3). \tag{15}$$

The corresponding sheaf of cubics must have the following equations, identical with (9*d*):

$$(c_2 - c_3)x_1^3 + (c_3 - c_1)x_2^3 + (c_1 - c_2)x_3^3 = 0.$$

Ad V, VI) From the cyclic structure of equations V and VI it is evident that they have as solutions the same sheaves of cubics that were found from II and III under the operation Θ. The values of λ are determined by inspection of the determinantal equation:

$$\begin{vmatrix} 2m\mu + 3\lambda & -2(m+\mu) & 2 \\ 2 & 2m\mu + 3\lambda & -2(m+\mu) \\ -2(m+\mu) & 2 & 2m\mu + 3\lambda \end{vmatrix} = 0; \tag{16}$$

$$\lambda = \tfrac{1}{3}\{2(m + \mu) - 2m\mu - 2\}, \tag{16a}$$

$$\lambda = \tfrac{1}{3}\{2(m + \mu)\omega - 2m\mu - 2\omega^2\}, \tag{16b}$$

$$\lambda = \tfrac{1}{3}\{2(m + \mu)\omega^2 - 2m\mu - 2\omega\}. \tag{16c}$$

Aside from two discrete cubics, which are not special covariants, the cubics invariant with respect to the transformation H *are four sheaves, identical with those left invariant by transformations* Θ *and* K.

§3. *Special Covariant Cubics whose Transformers are Irreducible.*

Mixed concomitants of order 3 and class 3 are four in number, beside the three reducible ones already discussed. Let each be denoted by G_i, where the suffix indicates the degree in the coefficients of the fundamental cubic. These four are readily calculated for the canonical form, from the fact that each is a Jacobian of simpler concomitants. Thus we may write:

$$\begin{aligned} G_3 &= J(\Theta, f, u_x) = 6\,(acu)\,(abu)^2 b_x c_x^2, \\ G_5 &= J(\mathrm{H}, f, u_x) = 6\,(abu) a_x b_x^2 a_s u_s^2, \\ G_7 &= J(\mathrm{K}, f, u_x), \\ G_9 &= J(\mathrm{K}, \Sigma, u_x), \end{aligned} \tag{17}$$

the Jacobian being formed in this last case by differentiating with respect to the variables (u_1, u_2, u_3) instead of (x_1, x_2, x_3).

The manner of employing these to transform point-loci is already explained; e. g., we define

$$G_3(L_x^3) = (acL)(abL)^2 b_x c_x^2, \text{ etc.} \tag{18}$$

The calculations involved in determining cubics invariant under these four transformations are so extensive that I will present here only those for the one of lowest degree, G_3. The omission of the others may be warranted also by this noteworthy fact:

The cubics which are invariant with respect to the transformers G_5, G_7, and G_9 respectively are identical with those which are invariant with respect to G_3, and consist of 8 discrete cubics and a sheaf, the syzygetic sheaf of the fundamental cubic and its Hessian.

The operator G_3.

Denoting by L_x^3 the cubic to be determined, and by λ an invariant factor, we impose the condition:

$$G_3(L_x^3) \equiv 6\lambda \cdot L_x^3.$$

The cubic being in the canonical form, we write the operator G_3 as in (17), when this condition becomes:

$$\begin{vmatrix} \left\{ \begin{matrix} 2\ (m^2L_1^2 + 2m\ L_2L_3)x_1 \\ -(\ \ L_2^2 + 2m^2L_3L_1)x_3 \\ -(\ \ L_3^2 + 2m^2L_2L_1)x_2 \end{matrix} \right\}, & x_1^2 + 2mx_2x_3, & L_1 \\ \left\{ \begin{matrix} 2\ (m^2L_2^2 + 2m\ L_3L_1)x_2 \\ -(\ \ L_3^2 + 2m^2L_1L_2)x_1 \\ -(\ \ L_1^2 + 2m^2L_3L_2)x_3 \end{matrix} \right\}, & x_2^2 + 2mx_3x_1, & L_2 \\ \left\{ \begin{matrix} 2\ (m^2L_3^2 + 2m\ L_1L_2)x_3 \\ -(\ \ L_1^2 + 2m^2L_2L_3)x_2 \\ -(\ \ L_2^2 + 2m^2L_1L_3)x_1 \end{matrix} \right\}, & x_3^2 + 2mx_1x_2, & L_3 \end{vmatrix} \equiv \lambda \cdot L_x^3. \tag{19}$$

Expanding this, we find the ten conditions, given typically by the following four:

$$-L_{222} + L_{333} = \lambda \cdot L_{111}, \quad \text{(3 equations)}; \tag{VII}$$

$$0 = 6\lambda \cdot L_{123}, \quad \text{(1 equation)}; \tag{VIII}$$

$$-(1 + 8m^3)L_{112} - 12m^2L_{223} - 6mL_{331} = 3\lambda \cdot L_{112}, \quad \text{(3 equations)}; \tag{IX}$$

$$(1 + 8m^3)L_{122} + 12m^2L_{311} + 6mL_{233} = 3\lambda \cdot L_{122}, \quad \text{(3 equations)}. \tag{X}$$

From (VII) we find by inspection the solution $\lambda = 0$, $L_{111} = L_{222} = L_{333}$; and combination gives the other values:

$$\lambda = \pm\sqrt{-3}.$$

The root $\lambda = 0$ occurs also in (VIII), so that we have a simply infinite sheaf:

$$(a) \qquad c_1(x_1^3 + x_2^3 + x_3^3) + c_2 \cdot x_1x_2x_3 = 0, \qquad (\lambda = 0),$$

the syzygetic sheaf of the fundamental cubic and its Hessian. The other two roots of (VII) give the two cubics:

$$(b_1) \qquad x_1^3 + \omega x_2^3 + \omega^2 x_3^3 = 0, \qquad (\lambda = -\omega + \omega^2);$$

$$(b_2) \qquad x_1^3 + \omega^2 x_2^3 + \omega x_3^3 = 0, \qquad (\lambda = -\omega^2 + \omega);$$

where ω is an imaginary cube root of unity.

Equations (IX) and (X) lead to the solutions, arranged alternately,

$$(c_1) \qquad x_1^2x_2 + x_2^2x_3 + x_3^2x_1 = 0, \qquad (\lambda = -\tfrac{1}{3}[1+2m]^3);$$

$$(c_2) \qquad x_1x_2^2 + x_2x_3^2 + x_3x_1^2 = 0, \qquad (\lambda = \tfrac{1}{3}[1+2m]^3);$$

$$(d_1) \qquad x_1^2x_2 + \omega x_2^2x_3 + \omega^2x_3^2x_1 = 0, \qquad (\lambda = -\tfrac{1}{3}[1+2\omega^2 m]^3);$$

$$(d_2) \qquad x_1^2x_3 + \omega x_2^2x_1 + \omega^2x_3^2x_2 = 0, \qquad (\lambda = \tfrac{1}{3}[1+2\omega^2 m]^3);$$

$$(e_1) \qquad x_1^2x_2 + \omega^2x_2^2x_3 + \omega x_3^2x_1 = 0, \qquad (\lambda = -\tfrac{1}{3}[1+2\omega m]^3);$$

$$(e_2) \qquad x_1^2x_3 + \omega^2x_2^2x_1 + \omega x_3^2x_2 = 0, \qquad (\lambda = \tfrac{1}{3}[1+2\omega m]^3).$$

For the transformations by G_5, G_7, and G_9, these invariant cubics reappear although the values of the invariant factors λ are of course altered, in so far as they are functions of the parameter m.* For G_5, for example, they are the following:

---

* For the sake of completeness I add the typical equations from which are determined the cubics invariant under the transformations symbolized by the mixed covariants G_7 and G_9 of degrees 7 and 9 in the coefficients of the fundamental cubic. It is understood that each set of ten equations is obtained from each of these sets of four by cyclical permutation of the indices 1, 2, 3.

From the condition: $G_7(L_x^3) = \lambda \cdot L_x^3$, one has the equations:

$$\begin{aligned}
\lambda L_1^3 &= -L_2^3 + L_3^3,\\
-3\lambda L_1^2L_2 &= (1+8m\mu^3)L_1^2L_2 + 4\mu(2m+\mu)L_2^2L_3 + 2(m+2\mu)L_3^2L_1,\\
3\lambda L_1L_2^2 &= (1+8m\mu^3)L_1L_2^2 + 4\mu(2m+\mu)L_2L_1^2 + 2(m+2\mu)L_2L_3^2,\\
6\lambda L_1L_2L_3 &= 0.
\end{aligned}$$

Here μ denotes the parameter of the Hessian.

From the condition: $G_9(L_x^3) = \lambda L_x^3$, one has the equations:

$$\begin{aligned}
\lambda L_1^3 &= -3m^3(5+4m^3)(L_2^3 - L_3^3),\\
3\lambda L_1^2L_2 &= (1-m^3)(1+4m^3)(1+8m^3)L_1^2L_2 + m(6+5m-60m^2-16m^4-16m^7)L_2^2L_3\\
&\qquad + (2+3m-20m^2-8m^3+80m^5+24m^7)L_3^2L_1,\\
-3\lambda L_1L_2^2 &= (1-m^3)(1+4m^3)(1+8m^3)L_1L_2^2 + m(6+5m-60m^2-16m^4-16m^7)L_3L_1^2\\
&\qquad + (2+3m-20m^2-8m^3+80m^5+24m^7)L_1L_2^2,\\
6\lambda L_1L_2L_3 &= 0.
\end{aligned}$$

$$(a) \qquad \lambda = 0\,,$$

$$(b_1) \qquad \lambda = -\omega + \omega^2\,,$$

$$(b_2) \qquad \lambda = -\omega^2 + \omega\,,$$

$$(c_1, c_2) \qquad 3\lambda = \mp\,[(1 + 8m^2\mu) + 4\ m\ (2\mu + m) + 2\ (\mu + 2m)]\,,$$

$$(d_1, d_2) \qquad 3\lambda = \mp\,[(1 + 8m^2\mu) + 4\omega m(2\mu + m) + 2\omega^2(\mu + 2m)]\,,$$

$$(e_1, e_2) \qquad 3\lambda = \mp\,[(1 + 8m^2\mu) + 4\omega^2 m(2\mu + m) + 2\omega(\mu + 2m)]\,,$$

(μ denoting the parameter of the Hessian).

It will be observed that these 8 cubics are particular cases of those found in sheaves as invariants of the transformations Θ, K, H, formulas (9*a*, 9*b*, 9*c*, 9*d*). *They are therefore invariants of every transformation whose symbol is compounded linearly from these seven:* Θ, K, H, G_3, G_5, G_7, G_9. *In other words, they are irrational combinants of the cubic and its Hessian.*

§4. *Geometrical Relations of the Irrational Covariant Cubics.*

For the canonical form of the cubic the triangle of reference is any one of the four inflexional triangles. The vertices of the inflexional triangles have accordingly the coördinates: *

$$\text{(I)} \quad \begin{matrix}(1 & 0 & 0)\\ (0 & 1 & 0)\\ (0 & 0 & 1)\end{matrix}, \qquad \text{(II)} \quad \begin{matrix}(1 & 1 & 1)\\ (1 & \omega & \omega^2)\\ (1 & \omega^2 & \omega)\end{matrix},$$

$$\text{(III)} \quad \begin{matrix}(1 & 1 & \omega)\\ (1 & \omega & 1)\\ (\omega & 1 & 1)\end{matrix}, \qquad \text{(IV)} \quad \begin{matrix}(1 & 1 & \omega^2)\\ (1 & \omega^2 & 1)\\ (\omega^2 & 1 & 1)\end{matrix}.$$

Inspection shows that each of the six cubics c_1, c_2, d_1, d_2, e_1, e_2, contains the three vertices of the inflexional triangle (I), and is tangent at each vertex to some one of its sides. The curves b_1 and b_2 do not contain those vertices, but do contain those of triangles (II), (III), and (IV). Further, the Hessian of b_1 and also of b_2 is the curve $x_1x_2x_3 = 0$, i. e., it is coincident with the triangle (I). Similar observations may be made upon the curves c_1, c_2 and triangle (II), the curves d_1, d_2 and triangle (III), and the curves e_1, e_2 and triangle (IV). Or the argument from symmetry may follow the first observations, giving us the theorem:

* See CLEBSCH-LINDEMANN, *Vorlesungen über Geometrie*, vol. I, p. 512.

Of the eight discrete covariant cubics which are invariant with respect to the seven transformers of point-loci here considered, each one is simultaneously inscribed and circumscribed to each of three of the inflexional triangles of the fundamental cubic, and has the fourth inflexional triangle for its Hessian. *

Since these covariant cubics are all equianharmonic, it would be an interesting problem to determine how many of the systems of three triangles which can be all simultaneously inscribed and circumscribed to an equianharmonic cubic can be inflexional triangles to another cubic curve.

§5. *The Seven Corresponding Transformers of Line Loci or Envelopes.*

A single general theorem will enable us to dispense with the detailed discussion of these seven operators in the dual rôle. Consider any operator of class 3 and order 3, $R^3_x u^3_\rho = R$. Employ it first as a transformer of point-loci, and denote the solutions of the invariant condition:

$$R^3_x L^3_\rho \equiv \mu \cdot L^3_x,$$

by $L_1, L_2, \cdots$, with factors $\mu_1, \mu_2, \cdots$. Next use it to transform line-loci, and denote the solutions of the defining equation:

$$R^3_\lambda u^3_\rho \equiv \nu \cdot u^3_\lambda,$$

by $\lambda_1, \lambda_2, \cdots$, with factors $\nu_1, \nu_2, \cdots$. *If μ_i is different from ν_κ, then L_i is apolar to λ_κ.* The proof is this: since

$$R^3_x L^3_\rho \equiv \mu L^3_x,$$

it follows that

$$R^3_\lambda L^3_\rho \equiv \mu \cdot L^3_\lambda;$$

and since

$$R^3_\lambda u^3_\rho \equiv \nu \cdot u^3_\lambda,$$

it follows that

$$R^3_\lambda L^3_\rho \equiv \nu \cdot L^3_\lambda;$$

whence by comparison, since $\mu \neq \nu$,

$$L^3_\lambda = 0;$$

and this is the expression for simple apolarity.

For each of the seven operators used, its matrical equation as a transformer of point-loci is identical with its matrical equation as a transformer of line-loci;

* How these same cubics may be found by considering the 18 collineations which leave unaltered the fundamental cubic may be seen from a paper in the Bulletin of the American Mathematical Society, vol. 4 (1897), p. 72.

the matrices are mutually transverse, and have the same set of roots, of which eight are discrete in the case of the irreducible cubic transformers. Hence we may draw the following conclusion:

The eight curves of class 3 which are invariant to all seven independent transformations here considered are each apolar to all cubics of the syzygetic sheaf, and to seven of the eight curves of order 3 discussed in §4.

This statement would enable us to write readily the equations of these eight line-cubics; but the further remark that all combinants of the syzygetic sheaf are combinants also of the conjugate sheaf of curves of class 3, and *vice versa*, while the inflexional triangles play the same rôle in both sheaves, leads to this theorem:

Those curves of class 3 which are invariant to every transformation of cubic line-loci whose symbol is a rational covariant of the fundamental cubic are a sheaf of line-cubics conjugate or doubly apolar to the corresponding members of the syzygetic sheaf of point-cubics, and, further, the eight curves of class 3 which are simultaneously circumscribed and inscribed, each to three out of the four inflexional triangles. Each of these eight line-cubics has the vertices of the fourth inflexional triangle for its Hessian and touches in nine points that one of the eight point-cubics to which it is not apolar.

NORTHWESTERN UNIVERSITY, EVANSTON, ILL.

A PURELY GEOMETRIC REPRESENTATION OF ALL POINTS IN THE PROJECTIVE PLANE*

BY

JULIAN LOWELL COOLIDGE

Introduction.

The advantage of having a basis of Geometric reality for the complex numbers of Algebra has long been apparent to mathematicians. More than two centuries ago JOHN WALLIS † devised a method of representing imaginary numbers by points in a plane which is truly ingenious, although hopelessly inferior to the well-known method with which we habitually associate the names of ARGAND and GAUSS. ‡ From another point of view, this latter method may be looked upon as a method of representing, by means of real points in a plane, the imaginary points of a real straight line.

The first mathematician to give a satisfactory purely geometric definition of imaginary elements was VON STAUDT, and in his *Beiträge zur Geometrie der Lage* § he shows how the GAUSS construction may be reached from a totally different point of view. He does not however outline his method in its most general form, as he makes use of the circular points at infinity, and a better statement has been given by KÖTTER. ‖ The same construction was used by HENRY J. S. SMITH ¶, but the end in view was different and he did not employ VON STAUDT'S distinction between conjugate imaginaries.

The problem of representing by means of real elements all points, not of a real straight line merely, but of a real plane, has not, to my knowledge, been

* Presented to the Society October 28, 1899, under a slightly different title. Received for publication March 12, 1900.

† WALLIS, *Algebra*, chapter 68.

‡ The true discoverer seems to have been WESSEL: *Om Direktionens analytiske Betegning*, Memoirs of Danish Academy, vol. 5, 1799. A French translation was laid before the same society by ZEUTHEN in 1897. A brief history of this whole subject will be found in *A Chapter in the History of Mathematics*, by W. W. BEMAN, Proceedings of the American Association, vol. 46, 1897.

§ VON STAUDT, *Beiträge zur Geometrie der Lage*, p. 264, Nuremberg 1856–1860.

‖ KÖTTER, *Grundzüge einer rein geometrischen Theorie der algebraischen ebenen Curven*, p. 21, Berlin 1887.

¶ *Mémoire sur Quelques Problèmes Cubiques et Biquadratiques*, Annali di Mathematica pura ed applicata, series 2, vol. 3, Milan 1869.

seriously attempted; or, at least, no purely geometric solution has been published. * One sees at once that the attempt to represent these points by the real points in space, to which one would naturally be led by the analogy of the method of GAUSS, must of necessity be fruitless, for the real points of space form but a triply infinite system, while the system of imaginary points of a real plane is quadruply infinite. We notice however that the system of real straight lines in space is quadruply infinite, and it is the object of this paper to show how they may be employed to represent the aforesaid system of imaginary points. After the necessary preliminary definitions of §1 I give in §2 a representation of all points in a real line by lines in a real plane † and then (§3) extend this representation so as to include all points in a real plane, noticing in particular those systems of lines which represent points on an imaginary line. I shall then (§§4, 5) take up the subject of chains of points, showing their application to the general theory of projectivity. After that (in §§6, 7) I shall glance briefly at the system of lines which represent points on a real conic, and conclude with a few remarks as to other possible solutions of our problem, and the extension to three dimensions.

§1. *Definitions.*

A linear range of points in elliptic involution, to which a particular sense of description is attached, is defined as an *imaginary point*. If the involution be looked upon as having the contrary sense, one has the *conjugate point*. The real line may be spoken of as the *base* of the point.

A linear pencil in elliptic involution is defined as the one or the other of two conjugate *imaginary lines of the first sort*, according as it is supposed to have the one or the other sense of description.

An analogous definition of an *imaginary plane* is obtained from an axial pencil in elliptic involution.

An *imaginary line of the second sort* is defined as the intersection of two imaginary planes whose real axes do not meet. This definition is from AUGUST ‡ and differs slightly in wording from that of VON STAUDT § though entirely equivalent thereunto in content. The imaginary line of the second sort differs from that of the first sort in that it contains no real point, and lies in no

*[Since this article went to press I have had brought to my notice that DUPORT in the Annales scientifiques de l'École Normale, ser. 2, vol. 9, p. 301, has given a very neat analytic solution of the problem of representing all points in a plane by real lines. (See the addition to §7.)]

† The construction appears as semi-correlative to that of KÖTTER. He establishes a correspondence of point to point by means of linear pencils; I shall establish a correspondence of point to line by means of projective ranges.

‡ *Untersuchungen über das Imaginäre in der Geometrie*; Programm der Friedrichsrealschule, Berlin 1872.

§ loc. cit., p. 77.

real plane. The system of all real lines meeting an imaginary line of the second sort (and also its conjugate, the intersection of the conjugate planes), that is, the bases of all points on such a line, form a congruence of the first order and class; one such line passes through every real point, and one lies in every real plane. All lines of the system which meet a given line, not of the system, are generators of a ruled quadric. No two lines of the system intersect.

§2. *Representation of Points of a Real Line.*

Let the real line be l, lying in a real plane π. Let a' be another real line in π cutting l in A and containing two other chosen real points O and Q. Finally let an imaginary line i' pass through Q. We may represent any point P of l by the real line or lines p through the intersection of OP with i'. If P be a real point distinct from A, p will coincide with OP and be unique. If P be the same as A, p will be any real line through Q. If P be imaginary, our statement of the construction has merely metaphorical significance, and the real geometric meaning must be examined more closely. Let i' be determined by the linear pencil in elliptic involution $a'a_1'$ $b'b_1'$, through Q, of which $a'a_1'$, $b'b_1'$ are two harmonically separated pairs, and $a'b'a_1'$ gives the sense of i'. In the same way let the imaginary point P be determined by the linear range AA_1BB_1 in elliptic involution, of which AA_1, BB_1 are harmonically separated pairs, and ABA_1 gives the sense P. The lines $O(BA_1B_1)$ may be called ba_1b_1 respectively. Since $a'ba_1b_1 \barwedge a'b'a_1'b_1'$ the lines bb', a_1a_1', and b_1b_1' will meet on a line p. The pencils $a'ba_1b_1$ and $a'b'a_1'b_1'$ will cut this line in pairs of the same involution, and to the sense $a'ba_1$ will correspond the sense $a'b'a_1'$. The line p will then be the base of the intersection of OP with i', and may be taken to represent P. Owing to the harmonic nature of the sets, $a'b'a_1'b_1' \barwedge a'b_1'a_1'b' \barwedge a'b_1a_1b$. The lines a_1a_1', bb_1', b_1b' will meet on a line p_1 bearing an involution of like sort to that on p with the exception that to the sense $a'ba_1$ will correspond the sense $a'b_1'a_1'$. The line p_1 will then represent P_1 the conjugate of P. It will be seen that pp_1 meet at the intersection of a_1a_1' and are harmonically divided by them.

Conversely, suppose that we have given a line p in π, and wish to find the corresponding point P on l. If p pass through O, P will be its intersection with l. If p pass through Q, P will coincide with A. If p pass through neither of these points, find its intersections with $b'a_1'b_1'$ and connect with O by lines ba_1b_1. The involution determined by $a'a_1$, bb_1 on l in sense $a'ba_1$ will be the point P required.

§3. *Representation of Points of a Real Plane.*

In representing all points of a real plane, we shall follow a method entirely analogous to that just explained for points on a line. Let the plane be called λ.

Let O be a real point and q a real line both in a plane a', neither however lying in the plane λ. Let an imaginary plane ι' pass through q. Any point P may be represented by the real line or lines p through the intersection of OP with ι'. If P be a real point not on the line $(a'\lambda)$, p will be the line OP. If P be a real point on $(a'\lambda)$, the lines p will be the bundle of lines through the intersection of OP with q.

If P be imaginary, we shall, as before, need a more elaborate construction. Let ι' be determined by an axial pencil in elliptic involution through q of which $a'a'_1$, $\beta'\beta'_1$ are two harmonic pairs and $a'\beta'a'_1$ gives the sense of ι'. Let P be an imaginary point of λ not on the line $(a'\lambda)$ but on some real line l. Let it be determined by an elliptic involution, whereof AA_1, BB_1 are harmonically separated pairs, A lying in $(a'\lambda)$, and ABA_1 giving the sense P. The plane Ol will cut ι' in an imaginary line i', and we shall represent P by the base of the intersection of OP and i', exactly as in two dimensions. If P be an imaginary point of the line $(a'\lambda)$ it will, in every case, be represented by the line q.

Conversely, for a real line p we find the corresponding point P. If p pass through O, the point sought is the intersection with λ. If p do not pass through O, nor yet meet the line q', we may confine ourselves to the plane Op and find P by the construction already explained for two dimensions. If p meet the line q, P will be the projection of this intersection from O on $(a'\lambda)$. If p coincides with q, P will be any point real or imaginary on $(a'\lambda)$.

Our scheme of correspondence is now complete, and, if we leave out the exceptional points on the line $(a'\lambda)$ which all correspond to the line q (and those which are real, to whole bundles concurrent on q besides), we see that to a single point corresponds a single real line, while to the points of a real line correspond the lines of a real plane through O, and conversely.

Let us now see how those lines lie which correspond to the points on an imaginary line (naturally, of the first sort) lying in our plane λ. Two important cases must be distinguished: (1) M the real point of the imaginary line i lies on $(a'\lambda)$; (2) M does not lie on $(a'\lambda)$. In the first case i will be determined in the sense $(a'\lambda)\,ba_1$ by an elliptic involution of lines, of which $(a'\lambda)a_1$, bb_1 are harmonically separated pairs. Let the planes $Ob\ Oa_1\ Ob_1$ be called $\beta a_1\beta_1$ respectively. Since $a'\beta'a'_1\beta'_1 \overline{\wedge} a'\beta a_1\beta_1$, while a' is self-corresponding, the intersections of $\beta'\beta$, a'_1a_1, $\beta'_1\beta_1$ will be three rays of a linear pencil in elliptic involution having its center at the point (OM, q) and lying in some plane μ. This pencil will be perspective both with $a'\beta'a'_1\beta'_1$ and with $a'\beta a_1\beta_1$, and to the sense $a'\beta'a'_1$ will correspond the sense $a'\beta a_1$. It will therefore determine an imaginary line k which is the intersection of planes Oi and ι'. Every point in i will thus be represented by the base p of a point in k, that is, by a real line in μ. Conversely, suppose that we have a real plane μ intersecting q. Connect the lines $(a'\mu)(\beta'\mu)(a'_1\mu)(\beta'_1\mu)$ with O by means of planes $a'\beta a_1\beta_1$. These planes will

cut λ in lines $(a'\lambda)ba_1b_1$ determining in sense $(a'\lambda)ba_1$ an imaginary line i whose real point M clearly lies on $(a'\lambda)$. To every line p in μ will correspond the intersection of the plane Op with the line i. It will be noticed that if μ pass through q the corresponding points will all line on $(a'\lambda)$.

Let us now take up the second case, in which M, the real point of i, does not lie on $(a'\lambda)$. Using the same notation as before and calling the plane $Oi\,\iota$ we see that the axes of the pencils $a'\beta'a_1'\beta_1'$ and $\beta a_1\beta_1$ do not meet and the intersections of the planes $\iota'\iota$ is an imaginary line of the second sort. The points of i are thus represented by the lines of a congruence. Any point of i, say that which lies on the real line l, is represented by that line of the congruence which lies in the plane Ol. The point M will be represented by the line OM, which is indeed the line of the congruence which passes through O. Conversely, an imaginary line k of the second sort lying in ι' will, with its conjugate, determine such a congruence of the first order and class. The congruence will determine about that ray which passes through O an axial pencil in elliptic involution, which will intersect λ in a linear pencil of the same sort. This latter, when taken in the sense corresponding to k, will give an imaginary line i, the locus of points represented by the lines of the congruence, since the plane Ok passes through ι'.

Before proceeding further let us recapitulate our results:

To each point in the plane and not on $(a'\lambda)$ will correspond a real line in space not meeting q, and conversely. To each real point on $(a'\lambda)$ will correspond the bundle of lines through its projection from O on q. To each imaginary point on $(a'\lambda)$ will correspond the single line q.

To all points on a real line will correspond all lines in the plane connecting it with O.

To all points on an imaginary line having its real point in $(a'\lambda)$ will correspond all lines in a plane, not through either O or q.

To all points on an imaginary line whose real point is not on $(a'\lambda)$ will correspond all lines of a congruence of first order and class determined by a certain imaginary line of the second sort in ι'.

It might seem as though we should also consider the problem of representing all points in an imaginary plane, but that need not detain us, for every point on the real axis may be represented by that axis, and every other point by the single real line on which it lies.

§4. *Chains of Points.*

It is now necessary to go into the difficult subject of chains of points. A few preliminary definitions will not be out of place. Let $pqrs$ be real bases of four points of an imaginary line of the second sort k. Let aba_1 be three lines meeting pqr, while aba_1 gives the sense of the points of k on the lines pqr. Three

possible cases may arise. In the first, s is a generator of the system pqr in the ruled quadric determined by the other six lines; in the second, the pencils $p(aba_1)$ and $a(pqr)$ cut s in ranges having the same sense, while, in the third, the ranges so cut have opposite senses. In the first case, the point (sk) is defined as having a sense neutral to that determined by the points (pk) (qk) (rk), in the second, as having the same or like sense, and in the third, as having the opposite sense. If s and t, the bases of two points of k, are separated by the quadric, the one will bear a point having the same sense as (pk) (qk) (rk) and the other a point having the opposite sense, * and conversely.

All those points of an imaginary line of the second sort whose sense is neutral to that determined by three given points of the line are said to belong to the *chain of points* determined by those three. The bases of all points of a chain are generators of a ruled quadric.

An exactly analogous treatment may be accorded to planes through an imaginary line of the second sort.

If $ABCD$ and $A_1B_1C_1D_1$ stand for two sets of points on, or planes through, an imaginary line of the second sort, the two groups of four points are said to be of the same sort, with regard to sense, if, according as D has a neutral, like, or opposite sense to that determined by ABC, D_1 has a neutral, like, or opposite sense to that determined by $A_1B_1C_1$, and in the first case if further to separated pairs separated pairs correspond.†

It may be shown by an elaborate investigation, which certainly need not be repeated here, that four points on a real line, or an imaginary one of the first sort, may be defined as having the same relation with regard to sense as the planes connecting them with an imaginary line of the second sort, and similarly for linear or axial pencils. Further, the relation of four elements with regard to sense is unchanged by an indefinite number of projections and intersections, real or imaginary.‡ If now we remember that an imaginary line in λ is represented, in the general case, by the real bases of points of an imaginary line of the second sort, we shall have:

The lines representing chains of points on an imaginary line in λ not having its real point on $(a'\lambda)$ will be generators of ruled quadrics. The lines representing points having the sense determined by three points will be separated by the quadric of the chain from the lines representing points having the opposite sense.

To investigate the system of lines representing chains of points on real lines in λ or on imaginary lines whose real points are on $(a'\lambda)$ it is necessary to look at

* Von Staudt, loc. cit., p. 43.

† This restriction is introduced in order to establish the continuous character of projective correspondence, and may be omitted where that is reached from other considerations. Compare Reye, *Geometrie der Lage*, vol. 1, p. 12 of Introduction, third edition, Leipzig, 1886.

‡ Compare Lüroth, *Das Imaginäre in der Geometrie und das Rechnen mit Wurfen*, Mathematische Annalen, vol. 8, pp. 170–172, 1875.

bases of chains of points on imaginary lines of the first sort. I am not aware that this has ever been done; it is quite easy after what we have already accomplished. For we see that an imaginary line of the first sort may easily be projected into one of the second, the bases of the points in the former becoming those of points in the latter, while all relations of sense remain unchanged. The generators of a ruled quadric will thus appear as the projection of the tangents to a real conic. As the imaginary line of the second sort lies in the quadric, so the line of the first sort will touch the conic, as will of course its conjugate. Two lines separated by the quadric will evidently appear as two lines, one of which cuts the conic in real points, while the other does not. This shows at once the arrangement of bases of points on an imaginary line of the first sort, and if we look upon the latter as arising from the intersection of the plane ι' with a real plane through O or with an imaginary one whose real axis meets q we shall have:

The lines representing chains of points on a real line in λ, or on an imaginary one whose real point is in $(a'\lambda)$ are tangents to a conic which touches ι', and its conjugate ι_1'. The systems of points whose sense is like to that determined by three chain points, and those whose sense is opposite, are represented by two systems of lines, one cutting the conic in real points, and the other in imaginary points.

§5. *General Theory of Projectivity.*

The question of chains is of particular importance because it leads to the general question of the projectivity of one-dimensional geometric forms, real or imaginary. The ordinary conception of projective figures is that of two figures which may be made the first and last of a sequence of perspective figures. VON STAUDT however defined* them as figures having a one-to-one correspondence and so related that four elements of one have the same relation with regard to sense as the four corresponding elements of the other and established * the important conclusion † that these two definitions amount to exactly the same. A proof of this theorem based on what we have here worked out may not be without interest.

As the relation of four elements with regard to sense is unaltered by projection, if two figures are the first and last of a sequence of perspective figures

* VON STAUDT, loc. cit., §§ 215-218, with the reference from § 218 to § 112 of the *Geometrie der Lage.*

† In the ordinary representation of a line on the Gauss plane a chain of points appears as a circle of points and the sense of D with respect to ABC depends upon the relation of D to the circle ABC with the sense ABC of description. VON STAUDT'S theorem implies that the one to one point transformations of the Gauss plane, which preserve circles and senses of point quadruples and the "projective" transformations:

$$w = \frac{az + \beta}{\gamma z + \delta},$$

of the complex variable are coextensive.

they will fulfil VON STAUDT'S definition. To show the converse, that two figures which answer to this definition may be made the first and last of a sequence of perspective figures, requires a rapid glance at harmonic forms.

The simplest linear constructions, such as connecting two points by a line, or passing a plane through a line and a point, may be performed unequivocally with imaginary elements as well as with real ones. If then we define four harmonic points by the familiar relation to a complete quadrangle, and four elements of a linear or axial pencil as harmonic if they project four harmonic points, the harmonic relation will be seen to be invariant in projection, whether the elements involved be real or imaginary. It may be shown further that four harmonic elements always belong to the same chain and that their bases appear in such form as four harmonic generators of a ruled quadric or four harmonic tangents to a conic. * In particular, four harmonic imaginary points on a real line in λ will be represented by four harmonic tangents to a conic in a plane through O, touching ι' and ι_1', and conversely.

Suppose now that we have two figures projective according to VON STAUDT'S definition: i. e., having a one-to-one correspondence, and the same relations of sense, among corresponding members. We may suppose them to be two real lines, as other cases may be reduced to this one by means of projections and sections. Consider the representing lines. To each chain-conic a^2 in one plane, will correspond a chain-conic a_1^2 in the other plane. To two chain-conics b^2 and c^2 tangent to a^2 will correspond two b_1^2 and c_1^2 touching a_1^2. To d^2 touching the real common tangents to b^2 and c^2 will correspond d_1^2 touching the tangents common to b_1^2 and c_1^2. Now it may easily be shown that the tangents common to a^2d^2 form a harmonic set with two tangents at points of contact of a^2b^2 and a^2c^2; similarly, in the other plane we shall have four harmonic tangents to a_1^2. † Hence, if our original figures are projective according to VON STAUDT'S definition, four harmonic elements of one correspond to four harmonic elements of the other. But the correspondence of harmonic elements, coupled with the correspondence of continuity and of sense gives, as is well known, the necessary and sufficient condition that two figures may be made the first and last of a sequence of projections and sections. Hence VON STAUDT'S definition is coextensive with the usual one. ‡

If we define the projectivity of a system of conics tangent to four given lines by that of the range of points of contact on one of these lines, they will also be

* LÜROTH, loc. cit., p. 176.

† Of course all conics here considered are chain-conics, and have two imaginary common tangents. An easy proof may be devised by considering the following special case of the correlative: If two circles touch a given circle, a circle through their intersection cuts the first in points harmonically separated by the points of contact.

‡ It is curious that the erroneous view that a one-to-one correspondence is necessarily a projective correspondence, has been widely held among mathematicians of prominence; compare:

projective with the system of tangents common to them and a fixed conic touching but three of the given lines. An easy proof of this may be obtained by transforming all the conics into points, by means of the transformation correlative to the generalized form of inversion.* This shows the correspondence of planes of lines corresponding to points of projective real linear ranges in λ, or imaginary ones with their real points in $(a'\lambda)$. To each line in one plane will in general correspond one line in the other, the lines which meet $(a'\lambda)$ being exceptions. To each set of tangents to a conic touching $\iota'\iota_1'$ in one will correspond a projective set of tangents to a conic of like sort in the other: to a set of such chain-conics tangent to two given lines in one will correspond a projective set tangent to the corresponding lines. Finally, to the set of lines cutting a chain-conic in real points in one plane will correspond either the set cutting the corresponding conic in real points, or that cutting it in imaginary ones. A criterion may be established for determining in any particular case in which of these two latter ways the correspondence will appear, but the labor involved is considerable, and seems scarcely worth while when we remember that the question settles itself if we know the positions of four lines in one plane corresponding to four lines not touching the same chain-conic in the other. The relation between these two planes is in general quite different from a real collineation, for a point appears as a degenerate chain-conic, and will correspond to a proper conic in the other.

§6. *Representation of Points on a Real Conic.*

It is now worth while briefly to consider the positions of those lines which represent points on a real conic in λ. We see at once that the lines representing real points on the conic are generators of a quadric cone with its vertex at O. As every real line in λ and every imaginary one having its real vertex in $(a'\lambda)$ cuts the conic twice, we have:

The lines representing points of a real conic in λ form a congruence of the second class. It will be seen that the same is true of the lines representing an

CHASLES, *Principe de correspondence entre deux objets variables, qui peut être d'un grand usage en Geometrie;* Comptes Rendus, vol. 41, Dec. 24, 1855.

CREMONA, *Introduzione ad una Teoria Geometrica delle Curve piane*, p. 7, Bologna 1862.

SALMON, *Conic Sections*, p. 304, sixth edition, London 1879.

A neat and very simple analytic proof of the absurdity of this view is given by GEISER, *Sopra un Teorema Fondamentale della Geometria;* Annali di Matematica, series 2, vol. 4, Milan 1870.

The necessity for insisting on the correspondence of sense, even after the correspondence of harmonic elements has been proved, may not be entirely evident. A simple example will make it more clear. Suppose that on any real line, each real point be supposed to correspond to itself, each imaginary point to its conjugate. We shall have a correspondence of chains, of harmonic points, of continuity, but not of sense; the correspondence is not projective.

* Compare HIRST, *On the quadric inversion of plane curves*, Proceedings of the Royal Society, vol. 14, p. 91, 1865.

imaginary conic, but we shall not take any further notice of this latter curve in the present article.

Let us assume that the conic has at least one real point A on $(a'\lambda)$. Take an imaginary point I of the curve, and connect it with the real points of $(a'\lambda)$. We shall have one chain of a linear pencil at I, and the points where these lines meet the conic again, as they form the whole system of points collinear with I and real points of $(a'\lambda)$, will be represented by all the lines of the system which meet the line representing I. Connecting these points with A, we shall have one chain of a pencil at A projective with one chain of a pencil at I. Now, since the chain of the pencil at A will cut any real line in a chain of points, which latter will be represented by tangents to a real conic, we see that the system of all points on the chain of the pencil at A will be represented by systems of lines lying in planes which pass through a common point, the projection of A on q, and cut a real plane through O in tangents to a conic; that is to say, the planes envelop a quadric cone. The planes tangent to this cone form a system projective with the axial pencil about the line representing I, and the intersections of the corresponding planes will be generators of a ruled cubic surface.* If the conic did not meet $(a'\lambda)$ in a real point, the lines of the system which met a chosen one of their number would still generate some surface. Moreover, that surface would be of the third order, for the conic in question might be obtained from the previous one by a homographic transformation of the plane, which, while altering the positions of the representing lines, would not alter the order of ruled surfaces determined by them. The general proposition is, then, that the lines representing points on a real conic, which meet a chosen line of the system, not representing a real point, are generators of a ruled cubic surface.

§ 7. *Other Possible Constructions.*

It is scarcely necessary to say that the method here described is by no means the only one which might have been used to represent the points of a real plane by means of real lines. For instance, we might have proceeded as follows: Suppose that we have given λ and two conjugate imaginary lines of the second sort ii_1. Each real point of λ may be represented by the real ray of the involution system ii_1 through it. Take an imaginary point ABA_1B_1 on a line l. The self-corresponding rays of the involution system which meet l will be generators of a ruled quadric forming the involution determined by ABA_1B_1. The generators of the other system will form the involution ii_1. As both involutions are elliptic, there will be one pair of self-corresponding lines which are axes of both.† The corresponding lines of the involution system obtained by

* Compare REYE, loc. cit., vol. 1, page 211.

† Compare VON STAUDT, loc. cit., p. 71.

joining criss-cross the intersections of these axes with λ and its mate λ_1, may be taken to represent ABA_1B_1 and its conjugate ABA_1B_1. It will not be difficult to find which line corresponds to which point. This method is peculiarly neat and symmetrical, for a real point is represented by a self-corresponding line while conjugate imaginaries appear as corresponding lines. It is, however, exceedingly difficult to manipulate, and for practical purposes inferior to the one which we have worked out.

[Still another method is the following. Let the plane λ be harmonically separated from a plane a by two planes $\beta\beta_1$. Let A be a chosen real point in a through which pass two conjugate imaginary lines ii_1. To represent a point P in λ, we connect the intersection of the real axis of the plane Pi with β and the intersection of the axis of the plane Pi_1 with β_1. This line is taken to represent the point P; if P be a real point the line will be PA. If a is the plane at infinity, and $\beta\beta_1$ are situated at a distance 1 from λ on either side, while A is the normal point to λ and ii_1 pass through the circular points at infinity in λ, we have the construction used by DUPORT in the paper cited in the Introduction. That paper is rich in results, and the algebraic expressions involved are particularly neat and symmetrical.]

The problem of representing all points in space by real figures does not offer a very favorable field for research. For, in order to employ straight lines in an unequivocal manner we should have to make use of a four-dimensional universe, no great help towards sense-perception. Again, we might use some other sextuply infinite system, as that of all twisted cubic curves through three given points, or all circles in space. But there would be a decided lack of naturalness about such a proceeding, nor would the results to be obtained appear in any degree commensurate with the extraordinary amount of labor involved.

HARVARD UNIVERSITY, CAMBRIDGE, MASS., *March* 10, 1900.

THE DECOMPOSITION OF THE GENERAL COLLINEATION OF SPACE INTO THREE SKEW REFLECTIONS*

BY

EDWIN B. WILSON

A number of years ago several investigators† published independently this theorem: *Any screw motion—the most general mechanical motion of space—can be decomposed into the product of two semi-rotations.* By a semi-rotation is meant that transformation which consists of rotating space through 180° about a fixed axis l. For generalizing however it is more convenient to look at this transformation as *reflection in a line.* From this point of view a semi-rotation may be defined as that transformation which replaces each point P of space by a point P' such that the line PP' cuts orthogonally a fixed line l and is bisected by it.

If we turn to the projective group of three dimensions we find in it, as a transformation corresponding to the semi-rotation of the mechanical group, the *skew reflection* which may be defined as follows: *A skew reflection is that transformation of space which replaces each point P by a point P' such that the line PP' cuts each of two non-coplanar lines l, l' and is divided harmonically by them.* The lines l, l' are called *directrices.*

The purpose of this paper is to ask and answer the question: *Is it possible to decompose the general collineation of space into the product of a number of skew reflections; and if so, what is the least number of skew reflections involved in such a decomposition?*‡ The question will be answered by giving

* Presented to the Society October 28, 1899. Received for publication November 18, 1899.

† HALPHEN, Nouvelles Annales de Mathématiques, 3d series, vol. 1, p. 296, 1882.
BURNSIDE, *On the Resultant of Two Finite Displacements of a Rigid Body;* Messenger of Mathematics, vol. 19, p. 104, 1889.
WIENER, *Die Zusammensetzung zweier endlicher Schraubungen zu einer einzigen;* Berichte d. Gesellschaft d. Wissenschaften zu Leipzig, vol. 42, p. 13, 1890.

‡ After writing this paper I was informed of an article by M. X. ANTOMARI entitled *Sur un cas particulier de la transformation homographique* in the Nouvelles Annales de Mathématiques, Nov., 1898, p. 489. In this paper M. ANTOMARI among other things decomposes the general collineation into the product of three special transformations which may be defined as follows: Each point P of space goes over into a point P' such that the line PP' cuts two given lines l, l' and is divided by them in a *given constant cross-ratio.* One sees that this transformation is *more general* than the skew reflection (in which the given constant cross-ratio is -1), and that it is *not involutory.* For these reasons the decomposition given by M. ANTOMARI is apparently of less interest and importance than the decomposition of this paper.

a geometric construction for the skew reflections whose product is the given collineation.

For purposes of proof one needs a number of preliminary propositions, corollaries of the definition of skew reflection.

I. Any line p cutting the two directrices l and l' of a skew reflection is transformed into itself.

II. The transformation of the points on p is such that any point P is replaced by such a point P' that P and P' divide harmonically the intersections of p with l and l'.

III. If a line p is transformed into itself it must cut both directrices, or coincide with one of them.

IV. Any skew reflection whose directrices cut harmonically two opposite edges of a tetrahedron merely interchanges the vertices and leaves the tetrahedron itself invariant.

V. If a skew reflection leaves a tetrahedron invariant and neither directrix contains a vertex of the tetrahedron, then the directrices cut harmonically two opposite edges of the tetrahedron.

LEMMA.—*There are ∞^2 skew reflections which interchange the vertices A and C, B and D of a given tetrahedron $ABCD$, leaving the tetrahedron itself invariant.*

For the two points of intersection of one of the directrices with two the opposite edges AC and BD of the tetrahedron may be chosen arbitrarily. The points of intersection of the other directrix with those edges are then uniquely determined by the harmonic property (IV). It is now evident that any given point P of space will be transformed into a point P', which will probably vary and describe a two-dimensional locus as the directrices of the skew-reflections, which leave the given tetrahedron invariant, take on their ∞^2 different positions.

THEOREM.—*The locus of the point P', into which a given general point P is carried by the ∞^2 skew reflections whose directrices cut harmonically two given opposite edges of a tetrahedron, is a quadric surface Q passing through and determined by the other four edges of the tetrahedron and the given point P.*

Let $ABCD$ be the given tetrahedron; AC and BD, its given opposite edges. We choose $ABCD$ as the coördinate tetrahedron $x_1x_2x_3x_4$. Let the directrix l meet AC in the point $(c_1, 0, c_3, 0)$ and BD in the point $(0, c_2, 0, c_4)$, where $c_1c_2c_3c_4 \neq 0$; the directrix l' then meets AC in the point $(c_1, 0, -c_3, 0)$ and BD in the point $(0, c_2, 0, -c_4)$. The skew reflection with directrices l, l' transforms a point P with the coördinates (y_1, y_2, y_3, y_4) into the point P' with the coördinates (y_1', y_2', y_3', y_4'), where

$$\rho y_1' = \frac{c_1}{c_3} y_3, \qquad \rho y_2' = \frac{c_2}{c_4} y_4, \qquad \rho y_3' = \frac{c_3}{c_1} y_1, \qquad \rho y_4' = \frac{c_4}{c_2} y_2,$$

as is readily proved. If the quantities c_1, c_2, c_3, c_4 are considered constant and the y's as the varying point coördinates, these equations establish the transformation of skew reflection between the points y and the points y'. If however we take the point $P, =(y_1, y_2, y_3, y_4)$, as any particular given point on no one of the faces of the tetrahedron, by eliminating the $c_1, \cdots, c_4, \rho$ (considered as variable) we have the locus of the transformed point $P', = (y'_1, y'_2, y'_3, y'_4)$, in the running coördinates x:

$$x_1x_3y_2y_4 = x_2x_4y_1y_3,$$

which is indeed a quadric surface through the point P and the four edges not cut by l and l':

$$x_1 = x_2 = 0, \qquad x_1 = x_3 = 0, \qquad x_2 = x_3 = 0, \qquad x_3 = x_4 = 0.$$

LEMMA.—*Two quadrics Q and Q', each of which passes through a different closed contour consisting of four edges of the same tetrahedron, intersect in at least one line which does not lie in any face of the tetrahedron.*

Given the tetrahedron $ABCD$. Let Q pass through the contour AB, BC, CD, DA; and Q' through AC, CB, BD, DA. Let P be any point on Q and in no face of the tetrahedron, and let P' be such a point on Q'. Draw the line p passing through P and cutting AB and DC. The line p has three points common with Q and hence lies entirely in Q. In like manner p', the line passing through P' and cutting AC and BD, will lie wholly in the quadric Q'. Since the three non-intersecting lines BC, p, and DA lie in Q, Q may be considered as generated by all lines cutting these three. In like manner Q' may be considered as generated by all lines cutting BC, p', and DA. But there is always at least one line which cuts any four lines. Hence there must be one line which cuts BC, DA, p, and p', and which consequently lies in both quadrics Q and Q'. Moreover it can not lie in any of the faces. For if it did, the points P and P' would lie in that face—which is contrary to hypothesis. Hence the lemma is established.

FUNDAMENTAL THEOREM.—*A collineation of space with four non-coplanar fixed points A, B, C, and D can be decomposed into the product of three skew reflections.**

Let the collineation be determined by its four fixed points A, B, C, D and a pair of corresponding points P and P'.

In the first take a skew reflection which has directrices cutting harmonically AC and BD and which carries P into some point $\bar{P}$ lying on the intersection

* It is to be noted that all cases of collineations which have more than four non-coplanar fixed points will come under this theorem. The cases where there are less than four require each a special treatment.

of the quadric Q, determined by P and the contour AB, BC, CD, DA, with the quadric Q' determined by P' and the contour AC, CB, BD, DA. Then take a skew reflection which has directrices cutting harmonically AB and DC and which carries P into P'. Finally take a skew reflection which has directrices cutting harmonically BC and DA and which leaves P' unchanged.

Then the product of these three skew reflections in the order named is the collineation which has A, B, C, D as fixed points and which carries P into P'. For by the first skew reflection A, B, C, D go over respectively into C, D, A, B; and these in turn by the second become respectively D, C, B, A; and these in their turn by the third go over respectively into the original positions A, B, C, D. Hence A, B, C, D are fixed points. By the construction also P was carried into P'. The three skew reflections thus give the required collineation.

That *two* skew reflections cannot be so chosen as to yield an arbitrarily given collineation may be seen at once from the following theorem, the proof of which involves no difficulty.

THEOREM.—*The necessary and sufficient condition that a collineation which has four and only four fixed points be resoluble into the product of two skew reflections is that the directrices of each of the two reflections shall cut harmonically one and the same pair of opposite edges of the fixed tetrahedron and that one pair of corresponding points lies on a quadric surface which passes through the other four edges of the tetrahedron.*

YALE UNIVERSITY,
October, 1899.

A NEW METHOD OF DETERMINING THE DIFFERENTIAL PARAMETERS AND INVARIANTS OF QUADRATIC DIFFERENTIAL QUANTICS*

BY

HEINRICH MASCHKE

In the following article I propose to exhibit in a preliminary way a symbolic method, in close analogy with the symbolism used in the algebraic theory of invariants, for the construction and investigation of invariants of quadratic differential quantics. The method proves to be fully as successful as in algebra, the chief advantage lying in the fact that after the establishment of the fundamental principles of the method further reference to the formulas of transformation becomes unnecessary.†

§1. *Introduction.*

If in the binary quadratic differential quantic

$$A = \sum_{i,k=1}^{2} a_{ik} dx_i dx_k, \qquad (a_{ki} = a_{ik}), \tag{1}$$

where the coefficients a_{ik} are functions of $x_1 x_2$, we introduce new variables $y_1 y_2$ by the equations:

$$x_i = x_i(y_1, y_2) \qquad (i = 1, 2), \tag{2}$$

then A goes into the new expression:

$$A = A' = \sum a'_{ik} dy_i dy_k. \tag{3}$$

Let now $u, v, \cdots$ be arbitrary functions of $x_1 x_2$, and denote by $u', v', \cdots$ the same functions after the substitution (2) has been made. Let further Ω be a function of the coefficients a_{ik} and their derivatives and of $u, v, \cdots$ and their derivatives, and let Ω' be the expression into which Ω is transformed, if the old quantities a_{ik}, $u, v, \cdots$ and their derivatives with respect to $x_1 x_2$ are replaced

* Presented to the Society (Chicago) April 14, 1900. Received for publication April 5, 1900.

† A similar remark applies to the method followed by Mr. HESSENBERG in his paper, *Über die Invarianten linearer und quadratischer binärer Differentialformen und ihre Anwendung auf die Deformation der Flächen*, Acta Mathematica, vol. 23, p. 121, 1900.

by the new quantities a'_{ik}, u', v', $\cdots$ and their derivatives with respect to $y_1 y_2$, where the a'_{ik} are defined by the equation (3). If Ω satisfies the equation:

$$\Omega' = \Omega, \tag{4}$$

Ω is called an *invariant of* the quadratic differential quantic A.

In this sense every arbitrary function $u, v, \cdots$ of $x_1 x_2$ is an invariant of A on account of the equations: $u' = u$, $v' = v$, $\cdots$.

Invariants Ω which actually contain at least one arbitrary function are generally called *differential parameters*, while those which involve no arbitrary function are called *differential invariants*.

The most important differential parameters are the following:*

$$\Delta_1 u = \sum_{r,s} A_{rs} \frac{\partial u}{\partial x_r} \frac{\partial u}{\partial x_s}, \tag{5}$$

$$\nabla(u, v) = \sum_{r,s} A_{rs} \frac{\partial u}{\partial x_r} \frac{\partial v}{\partial x_s}, \tag{6}$$

$$\Delta_2 u = \sum_{r,s} A_{rs} \frac{\partial^2 u}{\partial x_r \partial x_s} - \sum_{r,s,i,k} A_{rs} A_{ik} \begin{bmatrix} rs \\ k \end{bmatrix} \frac{\partial u}{\partial x_k}. \tag{7}$$

Here A_{ik} denotes the minor of the element a_{ik} in the determinant

$$D = |a_{ik}| \tag{8}$$

divided by the determinant D (which is always supposed to be different from zero). Further:

$$\begin{bmatrix} rs \\ k \end{bmatrix} = \tfrac{1}{2} \left(\frac{\partial a_{rk}}{\partial x_s} + \frac{\partial a_{sk}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_k} \right) \tag{9}$$

is the so-called *triple index symbol* introduced by CHRISTOFFEL.†

The transformed expression A' (3) being obtained from (1) by linear substitution of the differentials, the connection between the a'_{ik} and the a_{ik} will be the same as in the case of the transformation of an algebraic quantic with coefficients a_{ik} and the quantities $\partial x_i / \partial y_k$ as coefficients of the linear substitution.

Hence we have

$$D' = r^2 D, \tag{10}$$

where

$$r = \left| \frac{\partial x_i}{\partial y_k} \right| \qquad (i, k = 1, 2). \tag{11}$$

---

* Cf. BIANCHI, *Vorlesungen über Differentialgeometrie*, Leipzig, 1899, pp. 41 and 47.

† *Über die Transformation der homogenen Differentialausdrücke zweiten Grades*, Crelle's Journal, vol. 70, p. 48, formula (4).

If now F is an invariant of A, then

$$\frac{\partial F'}{\partial y_1}dy_1 + \frac{\partial F'}{\partial y_2}dy_2 = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2;$$

and if Φ is also an invariant, we have:

$$\frac{\partial F'}{\partial y_1}\frac{\partial \Phi'}{\partial y_2} - \frac{\partial F'}{\partial y_2}\frac{\partial \Phi'}{\partial y_1} = r\left(\frac{\partial F}{\partial x_1}\frac{\partial \Phi}{\partial x_2} - \frac{\partial F}{\partial x_2}\frac{\partial \Phi}{\partial x_1}\right). \tag{12}$$

With respect to any functions U and V of $x_1 x_2$ let us introduce the following abbreviations:

$$\frac{\partial U}{\partial x_i} = U_i, \qquad \frac{\partial^2 U}{\partial x_i \partial x_k} = U_{ik}, \quad \text{etc.}; \tag{13}$$

$$U_1 V_2 - U_2 V_1 = (U, V). \tag{14}$$

Further, with respect to a transformation from the variables $x_1 x_2$ to the variables $y_1 y_2$ we use the abbreviation:

$$\frac{1}{\sqrt{D}} = a. \tag{15}$$

Then from (12) and (10) we have in these notations:

$$a'(F, \Phi)' = a(F, \Phi),^* \tag{16}$$

and accordingly the fundamental theorem:

THEOREM I. *If F and Φ are invariants of A then $a(F, \Phi)$ is again an invariant of A.*

§2. *The Symbolic Method.*

Let now f be a (symbolic) function of $x_1 x_2$, and f_1 and f_2 its two derivatives; then on setting

$$f_i f_k = a_{ik}, \tag{17}$$

we may consider the expression $(f_1 dx_1 + f_2 dx_2)^2$ as symbolically equal to A. In this case we shall say for shortness that *f is a symbol of A.*

If $f, \phi, \cdots$ are several symbols of A, since $f' = f, \phi' = \phi, \cdots$, these symbols are in the first place invariants of A. Building up now according to theorem I other invariants out of these and of arbitrary functions $u, v, \cdots$ and forming products with these symbolic invariants as factors, we have at once by the same reasoning as in algebra an actual invariant of A before us, in case every

* [In subsequent papers it would be desirable to introduce for $a(F, \Phi)$ the notation $\{F, \Phi\}$.]

symbol $f, \phi, \cdots$ appears precisely twice and in such a way that only the connections $f_i f_k, \phi_i \phi_k, \cdots$ occur. (We shall presently see, however, that the last restriction can be partly removed.)

Thus the expression $a^2(f, u)^2$ satisfies the conditions just stated. Indeed we have

$$a^2(f, u)^2 = \frac{1}{D}(f_1^2 u_2^2 - 2f_1 f_2 u_1 u_2 + f_2^2 u_1^2) = \frac{1}{D}(a_{11}u_2^2 - 2a_{12}u_1 u_2 + a_{22}u_1^2)$$

$$= A_{11}u_1^2 + 2A_{12}u_1 u_2 + A_{22}u_2^2 = \sum A_{rs}\frac{\partial u}{\partial x_r}\frac{\partial u}{\partial x_s} = \Delta_1 u,$$

that is,

$$a^2(f, u)^2 = \Delta_1 u. \tag{18}$$

Likewise we obtain:

$$a^2(f, u)(f, v) = \nabla(u, v). \tag{19}$$

From (17) there follows:

$$\frac{\partial a_{ik}}{\partial x_l} = f_i f_{kl} + f_k f_{il},$$

and similarly:

$$\frac{\partial a_{kl}}{\partial x_i} = f_k f_{il} + f_i f_{kl},$$

$$\frac{\partial a_{il}}{\partial x_k} = f_i f_{ik} + f_i f_{ik}.$$

Hence we have the symbolic form:

$$f_i f_{kl} = \tfrac{1}{2}\left(\frac{\partial a_{ik}}{\partial x_l} + \frac{\partial a_{il}}{\partial x_k} - \frac{\partial a_{kl}}{\partial x_i}\right) = \begin{bmatrix} kl \\ i \end{bmatrix}, \tag{20}$$

of the triple index symbol.

We see therefore that, the other conditions given above being satisfied, we can also admit the connections $f_i f_{kl}$ as well as the connections $f_i f_k$.

For example, from theorem I we see that the expression $a\Big(f, a(f, u)\Big)$ is formally invariant; it also satisfies the conditions for actual existence. Indeed we find

$$a\Big(f, a(f, u)\Big) = a^2[f_1(f_{12}u_2 - f_{22}u_1) - f_2(f_{11}u_2 - f_{12}u_1)] + \cdots.$$

The complete computation leads to the result:

$$a\Big(f, a(f_1 u)\Big) = \Delta_2 u. \tag{21}$$

I give the values of some other symbolic expressions:

$$a^2(f, u)(\phi, u)\Big(f, a(\phi, u)\Big) = \tfrac{1}{2}\nabla(u, \Delta_1 u), \tag{22}$$

$$a^4(\phi, u)(\psi, u)\Big(f, a(\phi, u)\Big)\Big(f, a(\psi, u)\Big) = \tfrac{1}{4}\,\Delta_1(\Delta_1 u)\,, \tag{23}$$

$$a^2(f, \phi)\Big(a(f, u), a(\phi, u)\Big) = 2\Delta_{22}u\,, \text{*}$$

$$a^2\Big(f, a(\phi, u)\Big)\Big(\phi, a(f, u)\Big) = \Delta_2^2 u - 2\Delta_{22}u\,, \tag{24}$$

$$(f, \phi)\Big(f, a(\phi, u)\Big) = 0\,. \tag{25}$$

For the computation with symbolic expressions the following identity is useful:

$$(a, b)(c, d) + (a, c)(d, b) + (a, d)(b, c) = 0\,. \tag{26}$$

Also the relation (25) (which becomes an identity only when actual values are substituted for the symbols) often serves to reduce complicated expressions.

As an example the invariant $\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v)$ will be transformed. We have

$$\begin{aligned}\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v) &= a^4[(f, u)^2(\phi, v)^2 - (f, u)(f, v)(\phi, u)(\phi, v)] \\ &= a^4(f, u)(\phi, v)[(f, u)(\phi, v) - (f, v)(\phi, u)]\,.\end{aligned}$$

The quantity in the bracket reduces by means of (26) to $(f, \phi)(u, v)$. Hence

$$\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v) = a^4(f, \phi)(f, u)(\phi, v)(u, v)\,.$$

If to the right side of this equation we add the expression obtained from it by interchanging the equivalent symbols f, ϕ and divide by 2, we obtain:

$$\tfrac{1}{2}\,a^4(f, \phi)(u, v)\,[(f, u)(\phi, v) - (f, v)(\phi, u)].$$

Hence:

$$\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v) = \tfrac{1}{2}a^4(f, \phi)^2(u, v)^2 = \frac{1}{D}(u, v)^2\,.$$

As a second example of symbolic computation I give a proof of the formula:

$$2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) = 4\Delta_1 u \cdot \Delta_{22}u\,.\dagger$$

We have from (21), (22), (23):

$$2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) =$$
$$4a^4(\phi, u)\Big(f, a(\phi, u)\Big)\Big[(f, u)\Big(\psi, a(\psi, u)\Big) - (\psi, u)\Big(f, a(\psi, u)\Big)\Big]\,.$$

* For the definition of $\Delta_{22}u$ see BIANCHI, loc. cit., p. 48.

† This formula is given by BIANCHI, loc. cit., p. 48 without proof. Compare also HESSENBERG, loc. cit., p. 143.

The application of (26) reduces the quantity in the brackets to

$$(f, \psi)\left(u, a(\psi, u)\right).$$

Again by (26) we obtain:

$$\left(f, a(\phi, u)\right)\left(u, a(\psi, u)\right) = (u, f)\left(a(\psi, u), a(\phi, u)\right) + \left(f, a(\psi, u)\right)\left(u, a(\phi, u)\right).$$

The second term of this sum vanishes on account of (25) if we multiply by $(\phi, u)(f, \psi)$, and now we have:

$$2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) = -4a^4(f, u)(\phi, u)(f, \psi)\left(a(\psi, u), a(\phi, u)\right).$$

Interchanging here the equivalent symbols ϕ, ψ, adding the resulting expression to the right side, and dividing by 2, we obtain on the right side the factor $(\phi, u)(f, \psi) - (\psi, u)(f, \phi)$ which reduces, according to (26), to the product $(\phi, \psi)(f, u)$. The desired result is now obtained:

$$2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) = 2a^4(f, u)^2(\phi, \psi)\left(a(\phi, u), a(\psi, u)\right) = 4\Delta_1 u \cdot \Delta_{22} u.$$

The symbolic expressions for the second derivatives of a_{ik} appear as aggregates of the form $f_{ik}f_{lm}$ and $f_i f_{klm}$. But it is impossible to express conversely each of these two quantities separately in terms of the a_{ik} and their derivatives. There exist only certain connections of the $f_{ik}f_{lm}$ and $f_i f_{klm}$ which admit of such a representation, e. g.,

$$f_{11}f_{22} - f_{12}^2 = \tfrac{1}{2}\left(\frac{\partial^2 a_{11}}{\partial x_2^2} - 2\frac{\partial^2 a_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 a_{22}}{\partial x_1^2}\right),$$

and others. A similar remark holds good also for the case of all the higher derivatives of the a_{ik}.

The most important example of this kind is

$$a^2(\phi, \psi)\left(a(f, \phi), a(f, \psi)\right).$$

Derivatives of f of order k higher than the second do not occur, but the terms which are quadratic in the f_{ik} are all contained in the expression

$$\left((f, \phi), (f, \psi)\right),$$

an expression which can also be written in this form:

$$\frac{\partial(f, \phi)(f, \psi)_2}{\partial x_1} - \frac{\partial(f, \phi)(f, \psi)_1}{\partial x_2},$$

and here in the expressions which are to be differentiated, i. e., in $(f, \phi)(f, \psi)_1$ the coefficients f occur only in the connection $f_i f_{kl}$. Therefore the above symbolic invariant represents an actual invariant of A, and since it does not contain arbitrary functions, a differential invariant. The result is indeed:

$$a^2(\phi, \psi)\Big(a(f, \phi), a(f, \psi)\Big) = 2K,$$

where K is the expression* for *the curvature* of A. In passing we derive by comparison with (24) the formula:

$$K = \Delta_{22}(f).$$

§3. *Covariants.*

The complete differential of every invariant represents immediately a linear covariant of A. Covariants of higher degrees are therefore obtained at once by forming products of these symbolic linear covariants and (if necessary) also invariants, in such a way that every symbol $f, \phi, \cdots$ occurs precisely twice and so that the quadratic terms of the derivatives of the symbols admit of an actual interpretation in terms of the a_{ik} and their derivatives.

The simplest quadratic covariant is:

$$\left(\frac{\partial a(f, u)}{\partial x_1}\, dx_1 + \frac{\partial a(f, u)}{\partial x_2}\, dx_2\right)(f_1 dx_1 + f_2 dx_2).$$

This covariant is however irrational because the quantity a occurs only once as a factor. The simplest rational quadratic covariant is the following:

$$a(f, \phi)\left(\frac{\partial a(f, u)}{\partial x_1}\, dx_1 + \frac{\partial a(f, u)}{\partial x_2}\, dx_2\right)(\phi_1 dx_1 + \phi_2 dx_2);$$

it is identical with a covariant which BIANCHI† calls the second covariant differential of u.

As a further example I mention:

$$2a(f, \phi)\left(\frac{\partial a(f, F)}{\partial x_1}\, dx_1 + \frac{\partial a(f, F)}{\partial x_2}\, dx_2\right)(\phi_1 dx_1 + \phi_2 dx_2)(F_1 dx_1 + F_2 dx_2),$$

a cubic simultaneous covariant of A and $B = \sum b_{ik} dx_i dx_k$, where f, ϕ are symbols of A, and F is a symbol of B. This is one of the cubic covariants mentioned by BIANCHI, loc. cit., p. 55, and by KNOBLAUCH, loc. cit., p. 197.

* See BIANCHI, loc. cit., pp. 50–53, and KNOBLAUCH, *Einleitung in die Theorie der krummen Flächen*, Leipzig, 1888, pp. 174–179.

† loc. cit., p. 46.

§ 4. *Differential quantics in general.*

The preceding deductions can be extended without much difficulty from the case of binary differential quantics to those of n variables (and also from quadratic quantics to those of higher orders). So we have at once for ternary quantics:

$$\Delta_1 u = \tfrac{1}{2}\, a^2(f, \phi, u)^2,$$

$$\nabla(u, v) = \tfrac{1}{2}\, a^2(f, \phi, u)\,(f, \phi, v),$$

$$\Delta_2 u = \tfrac{1}{2}\, a\Big(f, \phi, a(f, \phi, u)\Big),$$

and here the generalization for n variables is evident.

The invariant forms which are analogous to the curvature are in the ternary case:

$$a^2(\phi, \psi, \chi)\,\Big(a(f, \phi, \psi),\, a(f, \chi, \omega),\, \omega\Big),$$

with f, ϕ, ψ, χ, ω as equivalent symbols, and in the general case:

$$a^2(\phi^1, \phi^2, \cdots \phi^n)\Big(a(f, \phi^1, \phi^2, \cdots \phi^{n-1}),\, a(f, \phi^n, \psi^1, \psi^2, \cdots \psi^{n-2}),\, \psi^1, \psi^2, \cdots \psi^{n-2}\Big),$$

with f, ϕ^1, ϕ^2, $\cdots$ ϕ^n, ψ^1, ψ^2, $\cdots$ ψ^{n-2} as equivalent symbols. It is easily seen that here the terms quadratic in f_{ik} occur only in the connection:

$$f_{il} f_{km} - f_{im} f_{kl} = \tfrac{1}{2}\left(\frac{\partial^2 a_{il}}{\partial x_k \partial x_m} + \frac{\partial^2 a_{km}}{\partial x_i \partial x_l} - \frac{\partial^2 a_{im}}{\partial x_i \partial x_l} - \frac{\partial^2 a_{kl}}{\partial x_i \partial x_m}\right).$$

THE UNIVERSITY OF CHICAGO, *April*, 1900.

ON THE EXTENSION OF DELAUNAY'S METHOD IN THE LUNAR THEORY TO THE GENERAL PROBLEM OF PLANETARY MOTION*

BY

G. W. HILL

PART I.—EXPOSITION OF THE THEORY.

The method of integrating the differential equations of motion, adopted by DELAUNAY for the elaboration of his lunar theory, as it is explained by him, demands its division into several cases, and is established through very tedious transformations. These disadvantages disappear when the greatest generality is given to the procedure. Hence, an explanation of the method, as it would be applied to the motion of a planetary system like the solar, will, doubtless, be welcome to astronomers.

I.

Let T denote the living force of the system, Ω the potential function, and, with POINCARÉ, put

$$F = \Omega - T.$$

The k variables, necessary for completely defining the position of the system, may be denoted by $q_1, q_2, \cdots, q_k$. Use accents to denote complete differentiation with respect to the time of the latter, and we have

$$T = \text{function}(q_1, q_2, \cdots, q_k, q_1', q_2', \cdots, q_k').$$

The partial derivatives of this function with respect to the k variables q_i' are to be used as variables instead of the latter, and we put

$$p_i = \frac{\partial T}{\partial q_i'}, \qquad (i = 1, 2, \cdots, k).$$

By means of these k equations the q_i' can be eliminated from T, and thus will result:

$$T = \text{function}(q_1, q_2, \cdots, q_k, p_1, p_2, \cdots, p_k).$$

* Presented to the Society February 24, 1900. Received for publication February 5, 1900.

Then the system of differential equations for determining the variables p_i and q_i is:

$$\frac{dp_i}{dt} = \frac{\partial F}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial F}{\partial p_i}, \qquad (i=1, 2, \cdots, k). \tag{1}$$

Let us suppose that Ω is separated into the two parts Ω_0 and Ω_1, and that, when we neglect Ω_1 in F, the equations (1) can be completely integrated, and their integrals expressed in terms of two sets of k quantities each, symbolized thus:

$$L_1, L_2, \cdots, L_k,$$
$$\lambda_1, \lambda_2, \cdots, \lambda_k,$$

of which the first set are constants, and the second set linear functions of the time of the form $n_i t + c_i$, n_i being a function of the L_i, and c_i an arbitrary constant. Nothing forbids our taking the L_i such that they may be the elements severally conjugate to the λ_i.

Now, desiring to integrate the equations (1) when F has its complete value, we may adopt the L_i and the λ_i as the dependent variables to be employed. The differential equations of the problem are then:

$$\frac{dL_i}{dt} = \frac{\partial F}{\partial \lambda_i}, \quad \frac{d\lambda_i}{dt} = -\frac{\partial F}{\partial L_i}, \qquad (i=1, 2, \cdots, k). \tag{2}$$

Here the function F has been made to involve the L_i and λ_i by eliminating the old variables p_i and q_i from it by means of their values given by the integrals derived on the supposition that $\Omega = \Omega_0$. As

$$F = \text{a constant}$$

is an integral of the problem, and $\Omega_0 - T = $ a constant, when Ω_1 is neglected, it is quite evident that when we substitute in $\Omega_0 - T$ for p_i and q_i their values in terms of the L_i and λ_i, the λ_i completely disappear and $\Omega_0 - T$ becomes a function of the L_i only. Thus, in the second form for F, the variables λ_i enter into it solely through the portion Ω_1.

II.

In order to exemplify we will adduce the solar system composed of the Sun and the eight major planets. We will suppose that the masses of the Sun, Mercury, Venus, $\cdots$, Neptune are denoted by m_0, m_1, m_2, $\cdots$, m_8, and will put

$$\mu_i = m_0 + m_1 + m_2 + \cdots + m_i, \quad \kappa_i = \frac{m_i}{\mu_i}, \qquad (i=0, 1, \cdots, 8).$$

Let the type of representation of the rectangular coördinates of the planets relative to the Sun be as follows:

$$\begin{aligned} &\text{Mercury} && x_1, \\ &\text{Venus} && x_2 + \kappa_1 x_1, \\ &\text{Earth} && x_3 + \kappa_2 x_2 + \kappa_1 x_1, \\ &\quad\cdots && \cdots\cdots\cdots \\ &\text{Neptune} && x_8 + \kappa_7 x_7 + \cdots + \kappa_1 x_1. \end{aligned}$$

The differential equations these variables satisfy are:

$$(3) \qquad \begin{cases} \mu_{i-1}\kappa_i \dfrac{d^2x_i}{dt^2} = \dfrac{\partial\Omega}{\partial x_i}, \\ \mu_{i-1}\kappa_i \dfrac{d^2y_i}{dt^2} = \dfrac{\partial\Omega}{\partial y_i}, \\ \mu_{i-1}\kappa_i \dfrac{d^2z_i}{dt^2} = \dfrac{\partial\Omega}{\partial z_i}, \end{cases} \qquad (i = 1, 2, \cdots, 8).$$

Here Ω denotes the sum of the products of every two masses of the system divided by their distance, a relation we will write thus:

$$(4) \qquad \Omega = m_0 \sum \frac{m_i}{\Delta_{0i}} + \sum \frac{m_i m_j}{\Delta_{ij}}.$$

Suppose that the portion to be separated from Ω is

$$(5) \qquad \Omega_0 = m_0 \sum \frac{m_i}{r_i},$$

r_i standing for $\sqrt{(x_i^2 + y_i^2 + z_i^2)}$. Then, if Ω_0 is substituted for Ω in equations (3), and the members are divided by $\mu_{i-1}\kappa_i$, we get

$$(6) \qquad \begin{cases} \dfrac{d^2x_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{x_i}{r_i^3} = 0, \\ \dfrac{d^2y_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{y_i}{r_i^3} = 0, \\ \dfrac{d^2z_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{z_i}{r_i^3} = 0, \end{cases} \qquad (i = 1, 2, \cdots, 8).$$

It will be seen that each group of these equations, corresponding to the same value of i, is independent of all the rest, and that it differs from the group of equations of relative motion of two bodies only in that the constant $m_0 \mu_i/\mu_{i-1}$ takes the place of $m_0 + m_i$.

Let a_i be the semi-axis major, e_i the eccentricity, ϕ_i the inclination, l_i the mean anomaly, g_i the angular distance of the perihelion from the node, and h_i the longitude of the node. Put

$$(7)\qquad \begin{cases} L_i = \sqrt{m_0 \dfrac{\mu_i}{\mu_{i-1}} a_i}\,, \\ G_i = L_i \sqrt{1 - e_i^2}\,, \\ H_i = G_i \cos \phi_i\,, \end{cases} \qquad (i = 1, 2, \cdots, 8).$$

Then, when the elements become variable by reason of the addition of Ω_1 to Ω_0, they will satisfy the differential equations:

$$(8)\qquad \begin{cases} \dfrac{dL_i}{dt} = \dfrac{\partial R_i}{\partial l_i}, \quad \dfrac{dl_i}{dt} = -\dfrac{\partial R_i}{\partial L_i}, \\ \dfrac{dG_i}{dt} = \dfrac{\partial R_i}{\partial g_i}, \quad \dfrac{dg_i}{dt} = -\dfrac{\partial R_i}{\partial G_i}, \\ \dfrac{dH_i}{dt} = \dfrac{\partial R_i}{\partial h_i}, \quad \dfrac{dh_i}{dt} = -\dfrac{dR_i}{\partial H_i}, \end{cases} \qquad (i = 1, 2, \cdots, 8),$$

where R_i will be, in terms of Ω_i, mentioned above,

$$(9)\qquad R_i = \frac{m_0 \dfrac{\mu_i}{\mu_{i-1}}}{2a_i} + \frac{\Omega_i}{\mu_{i-1}\kappa_i}, \qquad (i = 1, 2, \cdots, 8).$$

Desiring to have the same perturbative function, whatever may be the integer i, we multiply the values (7) of L_i, G_i, H_i, as also the value (9) of R_i by the constant $\mu_{i-1}\kappa_i$, which does not alter the form of equation (8). We now have:

$$(10)\qquad \begin{cases} L_i = m_i \sqrt{m_0 \dfrac{\mu_{i-1}}{\mu_i} a_i}\,, \\ G_i = L_i \sqrt{1 - e_i^2}\,, \\ H_i = G_i \cos \phi_i\,, \end{cases} \qquad (i = 1, 2, \cdots, 8),$$

as also:

$$(11)\qquad F = m_0 \sum_{i=1}^{i=8} \frac{m_i}{2a_i} + m_0 \sum_{i=2}^{i=\infty} m_i \left[\frac{1}{\Delta_{0,i}} - \frac{1}{r_i} \right] + \Sigma \frac{m_i m_j}{\Delta_{i,j}}.$$

If the planetary coördinates in the last equation are replaced by the elements (10) and the l_i, g_i, h_i, the differential equations of the system are:

$$(12)\qquad \begin{cases} \dfrac{dL_i}{dt} = \dfrac{\partial F}{\partial l_i}; & \dfrac{dl_i}{dt} = -\dfrac{\partial F}{\partial L_i}, \\ \dfrac{dG_i}{dt} = \dfrac{\partial F}{\partial g_i}, & \dfrac{dg_i}{dt} = -\dfrac{\partial F}{\partial G_i}, \\ \dfrac{dH_i}{dt} = \dfrac{\partial F}{\partial h_i}, & \dfrac{dh_i}{dt} = -\dfrac{\partial F}{\partial H_i}, \end{cases} \qquad (i = 1, 2, \cdots, 8).$$

In the second term of the right member of (11) the quantities $1/\Delta_{0,2}$, $1/\Delta_{0,3}$, $\cdots$, $1/\Delta_{0,8}$, can be developed in infinite series, the first terms of which are $1/r_2$, $1/r_3$, $\cdots$, $1/r_8$, and thus are cancelled by the term $1/r_i$. Then the two latter terms of (11) are of the second order with reference to planetary masses.

III.

In order to make the application of DELAUNAY'S method it appears necessary that F should be developed in a series, finite or infinite and periodic with respect to the variables l_i, g_i, h_i, which have been named the angular variables. In astronomical problems the series are generally infinite. For legitimate employment this series must remain convergent throughout the whole duration of motion, while t is passing from $-\infty$ to $+\infty$. It becomes then pertinent to ask what conditions must be fulfilled in order that this series may be convergent. It is well known that the reciprocal of the distance between two planets can be developed in a convergent infinite series, periodic with respect to the mean anomalies of the planets, provided that the orbits, as they stand in space, have no point in common, or when the reciprocal of the distance never becomes infinite. The condition of convergence in the present case is precisely similar to this. Here, however, not only the mean anomalies l_i are left indeterminate in the series, but also the remaining angular variables g_i and h_i which define the positions of the perihelia and nodes. Hence, in the present case, there must not only be no actual intersection of the orbits, but none when the perihelia and nodes are shifted in every possible way, the linear variables, or the mean distances, eccentricities and inclinations retaining their actual values. In the Delaunay development of the reciprocal of the distance between two planets, it is necessary and it suffices for convergence that the perihelion radius of one of the planets should always exceed the aphelion radius of the other.

We may consider this subject under a more general aspect. Let F have the periodic development

$$(13)\qquad F = \sum A \cos\left[j_1\lambda_1 + j_2\lambda_2 + \cdots + j_k\lambda_k\right],$$

where $\lambda_1 \cdots \lambda_k$ are the angular variables, the j positive or negative integers, and A is a function of the linear variables L only. That this infinite series may be

convergent, F must not only actually never become infinite, but never even potentially so. It is necessary here to explain what we mean by the qualifying epithet "potentially." If, while the linear variables L are supposed to maintain their actual values, and, consequently, the coefficients A their actual values, we allow to all the angular variables λ the complete swing of movement from 0 to 2π, F remains always finite, we say it never *potentially* becomes infinite. In order that F may not *actually* become infinite it is necessary and sufficient that the velocities of and the distances between the points of the system should remain finite. In order that F may not *potentially* become infinite, it is necessary and sufficient that the values of the linear variables $L_1, L_2, \cdots, L_k$ should remain within a certain domain. The definition of this domain is very complex after a Delaunay transformation has been operated, but is quite simple in terms of the original Keplerian linear variables $L_1, L_2, \cdots, L_k$.

We may illustrate this subject by bringing forward the case of the solar system as it has been described in § II. Employing the linear elements a_i and e_i of equation (10) or of equations (7), the inequalities which define this domain are:

$$(14)\qquad \begin{cases} a_1(1-e_1) & > 0, \\ a_2(1-e_2)-(1-\kappa_1)a_1(1+e_1) & > 0, \\ a_3(1-e_3)-(1-\kappa_2)a_2(1+e_2) & > 0, \\ \quad\cdots\cdots\cdots\cdots & \\ a_8(1-e_8)-(1-\kappa_7)a_8(1+e_7) & > 0, \\ a_8(1+e_8) & < \infty, \end{cases}$$

or, if we prefer to use the elements L_i and G_i of the same equations, the inequalities are:

$$(15)\qquad \begin{cases} L_1[L_1-\sqrt{L_1^2-G_1^2}] > 0, \\ \frac{\mu_2}{m_2^2}L_2[L_2-\sqrt{L_2^2-G_2^2}] > \frac{\mu_1}{m_1^2}L_1[L_1+\sqrt{L_1^2-G_1^2}], \\ \frac{\mu_3}{m_3^2}L_3[L_3-\sqrt{L_3^2-G_3^2}] > \frac{\mu_2}{m_2^2}L_2[L_2+\sqrt{L_2^2-G_2^2}], \\ \quad\cdots\cdots\cdots\cdots \\ \frac{\mu_8}{m_8^2}L_8[L_8-\sqrt{L_8^2-G_8^2}] > \frac{\mu_7}{m_7^2}L_7[L_7+\sqrt{L_7^2-G_7^2}], \\ L_8+\sqrt{L_8^2-G_8^2} < \infty. \end{cases}$$

It is remarkable that the inclinations of the orbits play no part in these inequalities.

IV.

As, in general, it is not necessary to distinguish between the three kinds l_i, g_i, h_i of angular variables, nor between the three kinds L_i, G_i, H_i, of linear variables, for simplicity of notation we shall suppose that the angular variables are denoted by $l_1, l_2, \cdots, l_k$, and their corresponding conjugate linear variables by $L_1, L_2, \cdots, L_k$.

Selecting a particular linear combination θ of of the angular variables so that

$$\theta = j_1 l_1 + j_2 l_2 + \cdots + j_k l_k,$$

the j being positive or negative integers prime to each other, Delaunay's method, somewhat generalized, consists in making such a transformation of variables as would constitute a complete solution of the problem if F, in its periodic development, contained as arguments only integral multiples of θ. That is, in this special case, the new linear variables would turn out constants, and the new angular variables would be of the form $n(t + c)$, n and c being likewise constants. It is clear that when we make such a transformation in F, the terms in the former periodic development involving the cosines of the finite multiples of θ will disappear, but the absolute term will receive a modification. A little consideration will make it evident that the derivation of such a transformation is dependent on quadratures.

The discussion of this derivation is greatly facilitated by making a linear transformation of variables, in the cases of both angular and linear variables. In this it is evident that we can take θ as one of the angular variables; then let Θ be its conjugate linear variable. Thus, we may have the following as the variables involved in the problem:

$$\text{Linear variables,} \quad \Theta, \Lambda_1, \Lambda_2, \cdots, \Lambda_{k-1};$$
$$\text{Angular variables,} \quad \theta, \lambda_1, \lambda_2, \cdots, \lambda_{k-1}.$$

And the canonical system of differential equations will be:

$$(16) \qquad \left\{ \begin{array}{llll} \dfrac{d\Theta}{dt} = \dfrac{\partial F}{\partial \theta}, & & \dfrac{d\theta}{dt} = -\dfrac{\partial F}{\partial \Theta}, & \\ \dfrac{d\Lambda_1}{dt} = \dfrac{\partial F}{\partial \lambda_1}, & & \dfrac{d\lambda_1}{dt} = -\dfrac{\partial F}{\partial \Lambda_1}, & \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{d\Lambda_{k-1}}{dt} = \dfrac{\partial F}{\partial \lambda_{k-1}}, & & \dfrac{d\lambda_{k-1}}{dt} = -\dfrac{\partial F}{\partial \Lambda_{k-1}}. & \end{array} \right.$$

Let us now consider the mean value of the function F relatively to the angular

variables $\lambda_1, \lambda_2, \cdots \lambda_{k-1}$. Then, since F as a periodic function involves only cosines of arguments, if $[F]$ denote the mentioned mean value, we shall have

$$[F] = \left[\frac{1}{\pi}\int_0^\pi\right]^{k-1} F d\lambda_1 d\lambda_2 \cdots d\lambda_{k-1}, \tag{17}$$

where the first factor of the right member denotes an operation repeated $k-1$ times, once in reference to each of the variables $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$. As F remains finite whatever values $\theta, \lambda_1, \lambda_2, \cdots, \lambda_k$ may assume, it follows that $[F]$ is finite whatever may be the value of θ. Thus $[F]$ is developable as a periodic function of θ involving only cosines; and we may write

$$[F] = -B - A_1 \cos\theta - A_2 \cos 2\theta - A_3 \cos 3\theta - \cdots, \tag{18}$$

where $B, A_1, A_2, A_3, \cdots$ are functions of the linear variables $\Theta, \Lambda_1, \Lambda_2, \cdots, \Lambda_{k-1}$.

Let us now suppose that, in equations (16), $[F]$ is substituted for F. They then become:

$$\left\{\begin{array}{llll} \dfrac{d\Theta}{dt} = \dfrac{\partial[F]}{\partial\theta}, & & \dfrac{d\theta}{dt} = -\dfrac{\partial[F]}{\partial\Theta}, \\ \dfrac{d\Lambda_1}{dt} = 0, & & \dfrac{d\lambda_1}{dt} = -\dfrac{\partial[F]}{\partial\Lambda_1}, \\ \cdots\cdots & & \cdots\cdots \\ \dfrac{d\Lambda_{k-1}}{dt} = 0, & & \dfrac{d\lambda_{k-1}}{dt} = -\dfrac{\partial[F]}{\partial\Lambda_{k-1}}. \end{array}\right. \tag{19}$$

$\Lambda_1, \Lambda_2, \cdots, \Lambda_{k-1}$ are therefore constants, and the two equations of the first line contain no other variables than Θ and θ, and thus form a distinct system by themselves and determine these two variables; after which, by substitution of values, the remaining differential equations for $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$ determine these variables through quadratures.

V.

As $[F]$ involves only two variables Θ and θ, the two equations which begin (19) have the integral, C being an arbitrary constant,

$$[F] + C = 0. \tag{20}$$

This integral constitutes a relation between the two variables Θ and θ; and, if the latter are regarded as coördinates defining the position of a point in a plane, (20) is the equation of a plane curve. For this graphical exhibition of the connection between the two variables, we might adopt that in which they are

the polar coördinates of a point, Θ being the radius and θ the angle. But, in some cases Θ may pass through zero. This difficulty may be obviated by adding to it a sufficiently large positive constant and thus it be rendered uniformly positive. This can be done provided it does not go to negative infinity. However, all circumstances considered, it will probably be a better course to adopt a representation in rectangular coördinates, θ being the abscissa and Θ the ordinate.

If we derive from (20) an expression for θ in terms of Θ and substitute it in the first equation of (19) and take the reciprocal of both members we shall have the time in terms of Θ by a quadrature, and, by the inversion of this, Θ as a function of t. On the other hand, if we derive from (20) an expression for Θ in terms of θ, and substitute it in the second equation of (19) we shall have the time in terms of θ by a quadrature, and, by the inversion of this, θ as a function of t.

We proceed to note some of the properties of the curve whose equation is (20). In the first place it must be stated that if the differential equations of (19), which determine the variables Θ and θ, compel the first of these to take on values rendering the right member of (18) a divergent series, we agree to set aside such cases as nugatory. Singularities of a certain kind are therefore excluded. The curve cannot have a *point d'arrêt*, for, at this point, we should have simultaneously $d\Theta/dt = 0$ and $d\theta/dt = 0$; and in consequence all succeeding derivatives of these variables would vanish. Thus, at this point Θ and θ would be invariable, which is impossible. It cannot have a multiple point, since, for given values of Θ and θ, there is but one value of each of the quantities $d\Theta/dt$ and $d\theta/dt$. If the curve pass through a point, it must proceed thence until it returns to that point or goes on to infinity. In the latter case, taking a polar representation for the moment, it may either have two infinite branches, or may make an infinite number of turns about the pole, or, in other words be a spiral. But, since equation (20) involves only cosines of θ without sines of the same, the curve must needs be symmetrically situated with respect to the axis from which θ is measured. Hence, the last supposition must be rejected; that is, it cannot be a spiral, nor can it have more than one distinct turn about the pole.

The curves graphically representing (20) may be divided into three classes. Here, for convenience, we adopt a rectangular representation. Let us suppose that an infinite number of values between the limits 0 and π are substituted for θ in (20); the result will be an infinite number of equations for determining the corresponding values of Θ. Let one of these be satisfied by a real value of Θ. Then it may happen that all the remaining equations are satisfied by real values of this variable continuous among themselves and with the value first mentioned. The variable θ can then move from $-\infty$ to $+\infty$ and there will always be a corresponding real value for Θ. The first equation of (19) shows that Θ will be at

a maximum or minimum when $\theta = i\pi$, i being a positive or negative integer. As, in the equation

$$\frac{d\Theta}{dt} = A_1 \sin\theta + 2A_2 \sin 2\theta + 3A_3 \sin 3\theta + \cdots, \tag{21}$$

the quantity A_1 is, in general, larger than $A_2, A_3, \cdots$, it follows that Θ will have no other maximum or minimum values than those just mentioned. In addition, if a maximum value occurs for $\theta = 2i\pi$, then will a minimum value occur for $\theta = (2i+1)\pi$, and *vice versa*. If, in (20) we put, in succession $\theta = 0$, $\theta = \pi$, we shall have the two equations:

$$\begin{cases} C - B = \quad A_1 + A_2 + A_3 + \cdots, \\ C - B = -A_1 + A_2 - A_3 + \cdots. \end{cases} \tag{22}$$

And if Θ be regarded as the unknown to be determined by them, it is plain that the maximum value of Θ will be a root of one of them and the minimum value a root of the other. Again Θ cannot be constant unless all the coefficients A vanish. It is quite evident that, in this case, the values of Θ and θ can be represented by the infinite periodic series:

$$\begin{cases} \Theta = \quad \Theta_0 \quad + \Theta_1 \cos[\theta_0(t+c)] + \Theta_2 \cos 2[\theta_0(t+c)] + \cdots, \\ \theta = \theta_0(t+c) + \theta_1 \sin[\theta_0(t+c)] + \theta_2 \sin 2[\theta_0(t+c)] + \cdots. \end{cases} \tag{23}$$

These two equations are to be regarded as the integrals of the first and second differential equations of the group (19); c is one of the arbitrary constants introduced by the integration, the other may be supposed to be either the C of (20) or the Θ_0 of the first of (23). But while C and c are conjugate to each other, this is not necessarily the case with the elements Θ_0 and $\theta_0(t+c)$. The remaining coefficients of (23), $\Theta_1, \Theta_2, \cdots, \theta_0, \theta_1, \theta_2, \cdots$, are functions of C or Θ_0. On account of the form of the curve which represents (20) in this case it may be called the sinusoid case.

We come now to consider the second case of the representation of (20) by a curve. Here, if we give to θ its range of values between 0 and π, we shall find that the equations determining the corresponding values of Θ have two real roots for an arc of values for θ which either begins at 0 or ends at π; and, in the first case the arc terminates, or, in the second case, begins, at the same intermediate point. At this point the two real roots become equal, and, for the remainder of the semi-circumference, they are imaginary. Consequently, at this point, θ attains either a maximum or minimum value. Because the equation contains only cosines of multiples of θ, in the one case, the right line $\theta = 0$, and, in the other, the right line $\theta = \pi$, divides the area embraced by the curve symmetrically. The maximum and minimum values of Θ are given by the roots of that

one of the two equations of (22) which has two real roots. In this case, θ can not be represented by series like the second of (23), but, in general, we may give the integrals of the problem the form:

$$(24)\qquad \begin{cases} \Theta\cos\theta = P_0 + P_1\cos[\theta_0(t+c)] + P_2\cos 2[\theta_0(t+c)] + \cdots, \\ \Theta\sin\theta = \quad Q_1\sin[\theta_0(t+c)] + Q_2\sin 2[\theta_0(t+c)] + \cdots, \end{cases}$$

where θ_0, P_0, P_1, P_2, $\cdots$, Q_1, Q_2, $\cdots$, are constant coefficients and functions of the C of (20), while c, as before, is the other arbitrary constant. It will be perceived that, in the former case, the integral equations (23) can be given the form (24) if one chooses; and DELAUNAY has always adopted it where the eccentricity e would appear as a divisor in the first form. At the two points, at which θ has attained its maximum or minimum value, we have $d\theta/dt = 0$, or

$$(25)\qquad \frac{dB}{d\Theta} + \frac{dA_1}{d\Theta}\cos\theta + \frac{dA_2}{d\Theta}\cos 2\theta + \cdots = 0.$$

When $dB/d\Theta$ and $dA_1/d\Theta$ are quantities of the same order of magnitude, the second case is likely to occur. As the curve, which here represents the connection of the variables Θ and θ, is a closed one, this case may be called the ovaloid case. This kind of motion in the variables is, however, generally termed a libration. Observation has not yet shown that it occurs in the system of the eight major planets of the solar system, although it is possible it may exist for very large values of the integers j_i. However, should this prove true, the influence of this circumstance on the motion of the system would be quite insignificant.

The third case in the graphical representation of (20) occurs when, in a certain range of values for θ, bisected by the value $\theta = 0$ or by the value $\theta = \pi$, we find a real value for Θ, but this value tends towards positive or negative infinity as the limits are approached. Here there is one maximum and no minimum for Θ or one minimum and no maximum. As in the previous cases, these values occur when $\theta = 0$ or $\theta = \pi$. As long as the instantaneous orbits of the planets composing the system are elliptic in their nature this case cannot present itself. And Θ cannot go beyond a certain limit without some of the elements becoming imaginary. In order, therefore, to prevent the occurrence of functions of complex variables, a modified system has to be adopted. But an illustration of this case can very easily be constructed. In order to escape the difficulty of divergence when $|\Theta|$ exceeds a certain limit, let us suppose that $[F']$ is finite and does not run into an infinite series, and that all the quantities A_i beyond A_1 vanish. Then the equation (20), being solved with reference to $\cos\theta$, gives

$$\cos\theta = \frac{C-B}{A_1} = \frac{f(\Theta)}{F(\Theta)}.$$

Let Θ_0 be the value of Θ when $\theta = 0$; in order to have the present case we ought to have

$$\frac{f(\Theta_0)}{F(\Theta_0)} = 1, \quad \frac{f(\infty)}{F(\infty)} = a,$$

a being less than unity. We may suppose that Θ is involved linearly in B and A_1, so that, a, b, c, d being constants,

$$f(\Theta) = a + b\Theta, \quad F(\Theta) = c + d\Theta.$$

Then

$$\Theta_0 = \frac{a-c}{d-b}, \quad a = \frac{b}{d}.$$

All the conditions will be fulfilled if we put

$$f(\Theta) = 3 + 4\Theta, \qquad F(\Theta) = 2 + 5\Theta;$$

whence $\Theta_0 = 1$ and $a = \frac{4}{5}$. Θ is thus continuous while θ is contained between the two values given by the equation $\cos\theta = \frac{4}{5}$. At the limits Θ becomes infinite. In a system of polar coördinates, if Θ is the radius and θ the angle, the equation of the curve graphically exhibiting the connection of the variables Θ and θ is:

$$\cos\theta = \frac{3 + 4\Theta}{2 + 5\Theta}, \text{ or } \Theta = \frac{-3 + 2\cos\theta}{4 - 5\cos\theta}.$$

It is thus a quartic algebraic curve whose equation in rectangular coördinates is:

$$[2x - 4(x^2 + y^2)]^2 = (3 - 5x)^2(x^2 + y^2),$$

whose course resembles that of a hyperbola. The formula for the time is:

$$t + c = \int \frac{d\Theta}{\sqrt{(2 + 5\Theta)^2 - (3 + 4\Theta)^2}} = \tfrac{1}{3}\int \frac{d\Theta}{\sqrt{(\Theta - \frac{2}{9})^2 - (\frac{7}{9})^2}}.$$

If this be integrated between the limits $\Theta = 1$ and $\Theta = \Theta$ it will give the time required to describe the curve from the point $\theta = 0$ to the point having the radius Θ.

VI.

Let us now suppose that by the integration of the system of differential equations (19), it is proposed to remove from F the periodic terms having the argument θ, that is, those contained in $[F]$. We confine ourselves to the first case as that will usually be the one which presents itself. The integrals of (19) will evidently have the form:

$$(26)\quad \begin{cases} \Theta &= \Theta_0 + \Theta_1 \cos\,[\theta_0(t+c)] + \Theta_2 \cos\,2[\theta_0(t+c)] + \cdots, \\ \Lambda_1 &= \text{a constant}, \quad \Lambda_2 = \text{a constant}, \quad \cdots, \quad \Lambda_{k-1} = \text{a constant}, \\ \theta &= \theta_0(t+c) + \theta_1 \sin\,[\theta_0(t+c)] + \theta_2 \sin\,2[\theta_0(t+c)] + \cdots, \\ \lambda_1 &= (\lambda_1) + \lambda_1^{(0)}(t+c) + \lambda_1^{(1)} \sin\,[\theta_0(t+c)] + \lambda_1^{(2)} \sin\,2[\theta_0(t+c)] + \cdots, \\ \lambda_2 &= (\lambda_2) + \lambda_2^{(0)}(t+c) + \lambda_2^{(1)} \sin\,[\theta_0(t+c)] + \lambda_2^{(2)} \sin\,2[\theta_0(t+c)] + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_{k-1} &= (\lambda_{k-1}) + \lambda_{k-1}^{(0)}(t+c) + \lambda_{k-1}^{(1)} \sin\,[\theta_0(t+c)] + \lambda_{k-1}^{(2)} \sin\,2[\theta_0(t+c)] + \cdots, \end{cases}$$

where c, (λ_1), (λ_2), $\cdots$, (λ_{k-1}), Λ_1, Λ_2, $\cdots$, Λ_{k-1}, and one other involved in $\Theta_0, \Theta_1, \cdots, \theta_0, \lambda_1^{(0)}, \lambda_1^{(1)}, \cdots$, complete the number of $2k$ arbitrary constants. If then we take formulas of transformation as follows, replacing

$$(27)\quad \begin{cases} \Theta & \text{by} \quad \Theta_0 + \Theta_1 \cos\theta + \Theta_2 \cos 2\theta + \cdots, \\ \Lambda_1 & \text{by} \quad \Lambda_1, \\ \cdots & \cdots\cdots \\ \Lambda_{k-1} & \text{by} \quad \Lambda_{k-1}, \\ \theta & \text{by} \quad \theta + \theta_1 \sin\theta + \theta_2 \sin 2\theta + \cdots, \\ \lambda_1 & \text{by} \quad \lambda_1 + \lambda_1^{(1)} \sin\theta + \lambda_1^{(2)} \sin 2\theta + \cdots, \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_{k-1} & \text{by} \quad \lambda_{k-1} + \lambda_{k-1}^{(1)} \sin\theta + \lambda_{k-1}^{(2)} \sin 2\theta + \cdots, \end{cases}$$

the terms of F involving θ as the argument will disappear from it, their mean values going over into the non-periodic term $-B$.

We shall now have a new set of angular variables θ, λ_1, λ_2, $\cdots$, λ_{k-1}, and it will be pertinent to inquire what are the linear variables conjugate to them, so that we may still have differential equations of the canonical form. Taking the two equations of (27) which define Θ and θ in terms of the new variables, let us suppose that the latter are Θ_0 and θ' (an accent is applied to the latter for the sake of discrimination). Then we have

$$(28)\quad \begin{cases} \Theta = \Theta_0 + \Theta_1 \cos\theta' + \Theta_2 \cos 2\theta' + \cdots, \\ \theta = \theta' + \theta_1 \sin\theta' + \theta_2 \sin 2\theta' + \cdots. \end{cases}$$

Then, by the substitution of these values in the two differential equations of (16) which determine Θ and θ, we get

$$(29)\quad \begin{cases} \dfrac{\partial\Theta}{\partial\Theta_0}\dfrac{d\Theta_0}{dt} + \dfrac{\partial\Theta}{\partial\theta'}\dfrac{d\theta'}{dt} = \dfrac{\partial F}{\partial\Theta_0}\dfrac{\partial\Theta_0}{\partial\theta} + \dfrac{\partial F}{\partial\theta'}\dfrac{\partial\theta'}{\partial\theta}, \\ \dfrac{\partial\theta}{\partial\Theta_0}\dfrac{d\Theta_0}{dt} + \dfrac{\partial\theta}{\partial\theta'}\dfrac{d\theta'}{dt} = -\dfrac{\partial F}{\partial\Theta_0}\dfrac{\partial\Theta_0}{\partial\Theta} - \dfrac{\partial F}{\partial\theta'}\dfrac{\partial\theta'}{\partial\Theta}. \end{cases}$$

By multiplying these equations by the proper factors, and putting Δ for the functional determinant or Jacobian:

$$\Delta = \frac{\partial \Theta}{\partial \Theta_0}\frac{\partial \theta}{\partial \theta'} - \frac{\partial \Theta}{\partial \theta'}\frac{\partial \theta}{\partial \Theta_0}, \tag{30}$$

we have

$$\Delta \frac{d\Theta_0}{dt} = \frac{\partial F}{\partial \theta'}, \quad \Delta \frac{d\theta'}{dt} = -\frac{\partial F}{\partial \Theta_0}. \tag{31}$$

But

$$\begin{aligned}\Delta = &\left[1 + \frac{\partial \Theta_1}{\partial \Theta_0}\cos\theta' + \frac{\partial \Theta_2}{\partial \Theta_0}\cos 2\theta' + \cdots\right]\left[1 + \theta_1\cos\theta' + 2\theta_2\cos 2\theta' + \cdots\right] \\ &+ \left[\Theta_1 \sin\theta' + 2\Theta_2 \sin 2\theta' + \cdots\right]\left[\frac{\partial \theta_1}{\partial \Theta_0}\sin\theta' + \frac{\partial \theta_2}{\partial \Theta_0}\sin 2\theta' + \cdots\right].\end{aligned} \tag{32}$$

According to the theorem of POISSON, Δ is independent of t, or what in this case amounts to the same thing, of θ'. Hence, in computing its value, we have regard only to the absolute term. Thus

$$\begin{aligned}\Delta &= 1 + \tfrac{1}{2}\left[\theta_1\frac{\partial \Theta_1}{\partial \Theta_0} + 2\theta_2\frac{\partial \Theta_2}{\partial \Theta_0} + \cdots + \Theta_1\frac{\partial \theta_1}{\partial \Theta_0} + 2\Theta_2\frac{\partial \theta_2}{\partial \Theta_0} + \cdots\right] \\ &= 1 + \tfrac{1}{2}\frac{\partial}{\partial \Theta_0}\left[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots\right].\end{aligned} \tag{33}$$

Then, if we adopt a new variable Θ' in place of Θ_0 such that

$$\Theta' = \int \Delta d\Theta_0 = \Theta_0 + \tfrac{1}{2}\left[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots\right], \tag{34}$$

equations (31) will be transformed into:

$$\frac{d\Theta'}{dt} = \frac{\partial F}{\partial \theta'}, \quad \frac{d\theta'}{dt} = -\frac{\partial F}{\partial \Theta'}, \tag{35}$$

which have the canonical form. As to the remaining linear variables $\Lambda_1, \Lambda_2, \cdots, \Lambda_{k-1}$, which are identical with the former variables denoted by the same symbols, it is evident that they remain the conjugates of the new variables $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$.

VII.

As it is somewhat difficult to discover the linear transformation of variables required to pass from the set

$$\left.\begin{matrix} L_1, & L_2, & \cdots, & L_k \\ l_1, & l_2, & \cdots, & l_k \end{matrix}\right\} \quad \text{to the set} \quad \left\{\begin{matrix} \Theta, & \Lambda_1, & \Lambda_2, & \cdots, & \Lambda_{k-1} \\ \theta, & \lambda_1, & \lambda_2, & \cdots, & \lambda_{k-1} \end{matrix}\right\},$$

suppose we adhere to the use of the first set. Then we have the differential equations

$$\frac{d\Theta}{dt} = \frac{\partial[F]}{\partial\theta}, \quad \frac{d\theta}{dt} = -\frac{\partial[F]}{\partial\Theta}, \tag{36}$$

$$\frac{dL_i}{dt} = j_i\frac{\partial[F]}{\partial\theta}, \quad \frac{dl_i}{dt} = -\frac{\partial[F]}{\partial L_i}, \qquad (i=1, 2, \cdots, k). \tag{37}$$

Then

$$L_i = (L_i) + j_i\Theta, \qquad (i=1, 2, \cdots, k), \tag{38}$$

where the (L_i) are arbitrary constants. By means of these equations the linear variables L_i may be eliminated from the right members of all the differential equations (36) and (37), and thus be replaced by the single variable Θ. Thus the mentioned right members become functions only of the two variables Θ and θ; and, after Θ and θ are found in terms of t from the integration of (36), the L_i will be found from (38), and the l_i from a quadrature operated on the second equations of (37).

Then we can suppose that the integrals of (36) and (37) are written thus:

$$\left\{\begin{aligned}
&\Theta = \Theta_1\cos[\theta_0(t+c)] + \Theta_2\cos 2[\theta_0(t+c)] + \cdots,\\
&L_1 = (L_1) + j_1\Theta_1\cos[\theta_0(t+c)] + j_1\Theta_2\cos 2[\theta_0(t+c)] + \cdots,\\
&\quad\cdots\cdots\cdots

The formulas of transformation are then:

$$(40)\qquad \begin{cases} \text{Replace} & L_1 & \text{by} & L_1 + j_1\Theta_1\cos\theta + j_1\Theta_2\cos 2\theta + \cdots, \\ & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \text{replace} & L_k & \text{by} & L_k + j_k\Theta_1\cos\theta + j_k\Theta_2\cos 2\theta + \cdots, \\ \text{replace} & l_1 & \text{by} & l_1 + l_1^{(1)}\sin\theta + l_1^{(2)}\sin 2\theta + \cdots, \\ & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \text{replace} & l_k & \text{by} & l_k + l_k^{(1)}\sin\theta + l_k^{(2)}\sin 2\theta + \cdots. \end{cases}$$

We have now to find what linear variables are conjugate to the new angular variables $l_1, l_2, \cdots l_k$. They are discovered immediately from a comparison of (34) and (38). As, from (34), it appears that Θ_0 the arbitrary constant, which may be conceived as annexed to the series for Θ, is not the element conjugate to the angular element $\theta_0(t+c)$, but that the expression

$$\tfrac{1}{2}[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots]$$

must be added to it to produce the required conjugate, it is plain from (38) that, after the transformation, the new L_i is no longer the exact conjugate of l_i, but that we have for that element the value

$$(41)\qquad L_i + \tfrac{1}{2}j_i[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots] \quad (i=1, 2, \cdots, k).$$

VIII.

In making one of DELAUNAY'S transformations it is not absolutely necessary that we should employ the linear variables L_i, which are the conjugates of the angular variables l_i, in the development of the various series needed; we may use any others connected with the former by known relations. Then equation (41) will inform us, at any stage of the transformations, what function the conjugates of the angular variables are of the used linear variables. Thus, in making his developments in the lunar theory, DELAUNAY has not used the elements he calls L, G, H, but has substituted for them others which he names a, e, γ.

Let us suppose that the new set of linear elements we determine to use are denoted by the symbols $e_1, e_2, \cdots, e_k$. Then, in order to form the equivalents for the dl_i/dt, it will be necessary to know the values of the partial derivatives $\partial e_i/\partial L_j$ in terms of the e_i. The number of these derivatives is k^2, and we shall have as many equations for determining them. Having the L_i in terms of the e_i, the general form of these equations will be:

$$(42)\qquad \frac{\partial L_i}{\partial e_1}\frac{\partial e_1}{\partial L_j} + \frac{\partial L_i}{\partial e_2}\frac{\partial e_2}{\partial L_j} + \cdots + \frac{\partial L_i}{\partial e_k}\frac{\partial e_k}{\partial L_j} = 0 \text{ or } 1,$$

according as i and j are different or are the same. These equations divide themselves into k groups each containing k equations; each group serving to determine the k partial derivatives of the e_i with reference to one of the L_i which we denote by L_j. The functional determinant of each group of equations is the same, being the Jacobian of the variables $L_1, L_2, \cdots, L_k$ with reference to the variables $e_1, e_2, \cdots, e_k$. Calling this determinant Δ, we shall have

$$\frac{\partial e_i}{\partial L_j} = \frac{1}{\Delta} \frac{\partial \Delta}{\partial \left(\frac{\partial L_j}{\partial e_i}\right)}. \tag{43}$$

We shall much abbreviate the calculation of these partial derivatives by agreeing to compute $\partial e_i/\partial L_j$ not directly but by a correction to be added to the value it had before the transformation was made. This correction will be denoted by $D(\partial e_i/\partial L_j)$. Now the transformation being that made to remove from F all the periodic terms having the argument

$$j_1 l_1 + j_2 l_2 + \cdots + j_k l_k,$$

suppose that, in this transformation, we put

$$K = \tfrac{1}{2}[\theta_1 \Theta_1 + 2\theta_2 \Theta_2 + 3\theta_3 \Theta_3 + \cdots], \tag{44}$$

and also

$$-K_m = \frac{\partial K}{\partial e_1}\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial K}{\partial e_2}\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial K}{\partial e_k}\left(\frac{\partial e_k}{\partial L_m}\right), \tag{45}$$

where $(\partial e_i/\partial L_m)$ denotes the value of the partial derivative before the transformation; then we shall have the group of linear equations:

$$\left\{\begin{aligned}
&\frac{\partial L_1}{\partial e_1} D\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial L_1}{\partial e_2} D\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial L_1}{\partial e_k} D\left(\frac{\partial e_k}{\partial L_m}\right) = j_1 K_m, \\
&\frac{\partial L_2}{\partial e_1} D\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial L_2}{\partial e_2} D\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial L_2}{\partial e_k} D\left(\frac{\partial e_k}{\partial L_m}\right) = j_2 K_m, \\
&\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&\frac{\partial L_k}{\partial e_1} D\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial L_k}{\partial e_2} D\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial L_k}{\partial e_k} D\left(\frac{\partial e_k}{\partial L_m}\right) = j_k K_m.
\end{aligned}\right. \tag{46}$$

Put now

$$P_i = j_1 \frac{\partial \Delta}{\partial \left(\frac{\partial L_1}{\partial e_i}\right)} + j_2 \frac{\partial \Delta}{\partial \left(\frac{\partial L_2}{\partial e_i}\right)} + \cdots + j_k \frac{\partial \Delta}{\partial \left(\frac{\partial L_k}{\partial e_i}\right)}. \tag{47}$$

Then, evidently,

$$D\left(\frac{\partial e_i}{\partial L_m}\right) = \frac{P_i K_m}{\Delta}. \tag{48}$$

Thus the corrections for each partial derivative, caused by a given transformation, are reduced to the product of three factors, one of which is a function of i, another a function of m, and the third independent of either. The elements e_i, before the performance of any transformation, may be what we choose. DELAUNAY chose his a, e, γ so that they were connected with L, G, H, the conjugates of the angular variables l, g, h, by the relations:

$$L = \sqrt{\mu a}, \qquad G = L\sqrt{1-e^2}, \qquad H = G(1-2\gamma^2).$$

PART II.

APPLICATION OF DELAUNAY'S METHOD TO THE MINOR PLANET OF THE HECUBA TYPE.

IX.

DELAUNAY'S lunar theory affords a plentiful assortment of the transformations just discussed, but their application in a case of planetary motion gives rise to more complex expressions. In the lunar theory it is possible to expand all coefficients in power series of all the parameters involved; but, in a planetary theory where a, the ratio of the mean distances, is a considerable fraction, it is necessary to introduce the functions of a usually denoted by the symbol $b_s^{(i)}$, as also their derivatives with respect to a. It may therefore be profitable to give as simple an illustration as possible of these transformations where $b_s^{(i)}$ must be used.

Let Jupiter be supposed to describe a circular orbit about the Sun, while a small planet, without mass, describes an orbit in the same plane. Let the radius of Jupiter's orbit be taken as the linear unit; denote its longitude by $\epsilon' + n't$, and the masses severally of the Sun and Jupiter by m_0 and m'. Let a, e, l, and g be the mean distance, eccentricity, mean anomaly, and longitude of the perihelion of the small planet, and r and v its radius and true anomaly. Put

$$\gamma = g - \epsilon' - n't, \qquad L = \sqrt{m_0 a}, \qquad \Gamma = \sqrt{m_0 a(1-e^2)}.$$

The function we have denoted by F will have the expression:

$$(49) \quad F = \frac{m_0}{2a} + n'\Gamma + m'\{[1 - 2r\cos(v+\gamma) + r^2]^{-\frac{1}{2}} - r\cos(v+\gamma)\},$$

where r and v are to be eliminated through the equations:

$$r\cos v = a(\cos u - e), \quad r\sin v = a\sqrt{1-e^2}\sin u, \quad u - e\sin u = l.$$

The position of the small planet will be known when we know L, Γ, l, γ.

The differential equations for determining the latter are:

$$(50) \qquad \begin{cases} \dfrac{dL}{dt} = \dfrac{\partial F}{\partial l}, & \dfrac{dl}{dt} = -\dfrac{\partial F}{\partial L}, \\ \dfrac{d\Gamma}{dt} = \dfrac{\partial F}{\partial \gamma}, & \dfrac{d\gamma}{dt} = -\dfrac{\partial F}{\partial \Gamma}. \end{cases}$$

X.

In order to give an illustration of the transformations named by DELAUNAY *operations*, let us select from the periodic development of F, which, from (49), plainly has the form $\sum A_{i,i'} \cos(il + i'\gamma)$, all the terms having the argument $\theta = l + 2\gamma$. These will be terms of long period in case the small planet has a mean motion nearly double that of Jupiter, which case has been extensively discussed by astronomers, such a minor planet being called of the Hecuba type. Taking θ as one of the angular elements, we see that we can adopt γ as the other, and thus shall have $l = \theta - 2\gamma$. In order to obtain $[F]$ from F we have the equation:

$$(51) \qquad [F] = \frac{m_0}{2a} + n'\Gamma + \frac{1}{\pi}\int_0^\pi \frac{m'\,d\gamma}{\sqrt{1 - 2r\cos(v+\gamma) + r^2}},$$

remembering that r and v are now the same functions of $\theta - 2\gamma$ they were before of l. The last term of F in (49) is here omitted as it contributes nothing to $[F]$. Put

$$(52) \qquad [1 - 2r\cos(v+\gamma) + r^2]^{-\frac{1}{2}} = \tfrac{1}{2}B^{(0)} + B^{(1)}\cos(v+\gamma) + B^{(2)}\cos 2(v+\gamma) + \cdots,$$

where $B^{(i)}$ is the same function of r that $b_{\frac{1}{2}}^{(i)}$ is of a. In order that this series may be convergent it is necessary that $a(1+e) < 1$. Let us put

$$(53) \qquad A^{(i)} = \frac{1}{\pi}\int_0^\pi B^{(2i)}\cos i(2v - l)\,dl.$$

Then we have

$$(54) \qquad [F] = \frac{m_0}{2a} + n'\sqrt{m_0 a(1-e^2)} + m'\left[\tfrac{1}{2}A^{(0)} + A^{(1)}\cos\theta + A^{(2)}\cos 2\theta + \cdots\right].$$

The investigation will be facilitated if we now make a slight change in the dependent variables employed so that they have the following equivalents:

$$(55) \qquad \begin{cases} \Theta = \sqrt{m_0 a}, & \Gamma = \sqrt{m_0 a}\,[2 - \sqrt{1-e^2}], \\ \theta = l + 2g - 2\epsilon' - 2n't, & \gamma = \epsilon' + n't - g. \end{cases}$$

Then the differential equations determining the formulas of transformation are:

$$(56)\qquad \begin{cases} \dfrac{d\Theta}{dt} = \dfrac{\partial[F]}{\partial\theta}, & \dfrac{d\theta}{dt} = -\dfrac{\partial[F]}{\partial\Theta}, \\ \dfrac{d\Gamma}{dt} = 0, & \dfrac{d\gamma}{dt} = -\dfrac{\partial[F]}{\partial\Gamma}. \end{cases}$$

Of these equations the integral of the third, $\Gamma =$ a constant, furnishes the relation:

$$(57)\qquad a = a[2 - \sqrt{1-e^2}]^{-2} = a[1 - e^2 + \tfrac{1}{2}e^4 - \tfrac{1}{4}e^6 + \tfrac{3}{32}e^8 - \cdots],$$

a being a constant. By means of this relation the variable a may be eliminated from $[F]$ which thus will contain but two variables, e and θ. The equations (56) have the canonical form, but we prefer to discard the variable Γ and to use e in its stead. Supposing then that $[F]$ is made a function of the variables e and θ, the differential equations for the latter are:

$$(58)\qquad \begin{cases} \dfrac{de}{dt} = -\dfrac{1}{\sqrt{m_0 a}} \dfrac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]^2}{e} \dfrac{\partial[F]}{\partial\theta}, \\ \dfrac{d\theta}{dt} = \dfrac{1}{\sqrt{m_0 a}} \dfrac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]^2}{e} \dfrac{\partial[F]}{\partial e}, \end{cases}$$

where the factor

$$\frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]^2}{e} = \frac{1}{e}[1 + \tfrac{1}{2}e^2 - \tfrac{1}{8}e^4 - \tfrac{3}{16}e^6 - \tfrac{17}{128}e^8 - \cdots].$$

These equations form a group to be integrated by themselves. After this integration is accomplished, γ is derived through a quadrature of the equation:

$$\frac{d\gamma}{dt} = -\frac{\partial[F]}{\partial\Gamma}.$$

In this equation $[F]$ is a function of Θ and Γ, but we have preferred to write it as a function of a and e; thus:

$$\frac{\partial[F]}{\partial\Gamma} = \frac{\partial[F]}{\partial a}\frac{\partial a}{\partial\Gamma} + \frac{\partial[F]}{\partial e}\frac{\partial e}{\partial\Gamma}.$$

But

$$\Theta = \frac{\sqrt{m_0 a}}{2-\sqrt{1-e^2}}, \qquad \Gamma = \sqrt{m_0 a},$$

$$a = \frac{\Gamma^2}{m_0}, \qquad \sqrt{1-e^2} = 2 - \frac{\Gamma}{\Theta}.$$

Consequently

$$\frac{\partial a}{\partial \Gamma} = 2\sqrt{\frac{a}{m_0}}, \qquad \frac{\partial e}{\partial \Gamma} = \frac{1}{\sqrt{m_0 a}} \frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]}{e}.$$

Remembering that, with our adopted linear unit, $a' = 1$, $n' = \sqrt{m_0 + m'}$, we have:

$$\frac{1}{n'}\frac{d\gamma}{dt} = -\frac{1}{m_0\sqrt{\left(1+\frac{m'}{m_0}\right)a}}\left[2a\frac{\partial[F]}{\partial a} + \frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]}{e}\frac{\partial[F]}{\partial e}\right].$$

But, adopting, as before, g for the longitude of the perihelion, this is more simply:

$$\frac{1}{n'}\frac{dg}{dt} = \frac{1}{\sqrt{\left(1+\frac{m'}{m_0}\right)a}}\left[2a\frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]}{e}\frac{\partial R}{\partial e}\right]$$

$$= \frac{2}{\sqrt{\left(1+\frac{m'}{m_0}\right)a}}\, a\frac{\partial R}{\partial a} + \frac{1}{2-\sqrt{1-e^2}}\frac{1}{n'}\frac{d\theta}{dt}$$

(59)
$$- \frac{1}{\sqrt{\left(1+\frac{m'}{m_0}\right)a}}[2-\sqrt{1-e^2}]^2 + \frac{2}{2-\sqrt{1-e^2}},$$

where

$$R = \frac{m'}{m_0}\frac{1}{\pi}\int_0^\pi \frac{d\gamma}{\sqrt{1-2r\cos(v-\gamma)+r^2}}.$$

XI.

In an application like the present, where the periodic developments of the various quantities are always tardily convergent, it is nearly impossible to give literal expressions for the coefficients. And, even if we consent to give to each coefficient its numerical value at once, the work of multiplying such periodic series together is very embarrassing, and the process easily leads to the commission of errors. Hence we adopt the method of substituting for each quantity involved the special values of it at equal intervals in the motion of the independent variable through the semicircumference. With this method of treatment it is necessary to separate the cases of non-libration and libration.

It is always an advantage in computation to have the members dealt with independent of any linear and temporal units. To this end let us substitute for

the independent variable t, the variable $\tau = \epsilon' + n't$ or the longitude of Jupiter; also we put

$$W = \frac{[F]}{m_0}, \quad \nu = \frac{m'}{m_0}.$$

The coefficients of the periodic development of W are then absolute numbers. The equations which, with (57), we shall use for the elaboration of the problem, are the three following equations:

$$(60) \quad \begin{cases} W = \text{a constant}, \\ \dfrac{d\tau}{d\theta} = \sqrt{(1+\nu)\,a}\,\dfrac{e}{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]^2}\,\dfrac{1}{\dfrac{\partial W}{\partial e}}, \\ \dfrac{dg}{d\tau} = \dfrac{2}{\sqrt{(1+\nu)\,a}}\,a\,\dfrac{\partial R}{\partial a} + \dfrac{1}{2-\sqrt{1-e^2}}\,\dfrac{d\theta}{d\tau} \\ \qquad\qquad - \dfrac{1}{\sqrt{(1+\nu)\,a}}\,[2-\sqrt{1-e^2}]^2 + \dfrac{2}{2-\sqrt{1-e^2}}; \end{cases}$$

W has the expression:

$$(61) \quad \begin{aligned} W &= \frac{1}{2a}\,[2-\sqrt{1-e^2}]^2 + \sqrt{(1+\nu)\,a}\,\frac{1-e^2+2\sqrt{1-e^2}}{3+e^2} + \nu\,\frac{1}{\pi}\int_0^\pi \frac{d\gamma}{\Delta} \\ &= \frac{1}{2a}\,[1 + e^2 + \tfrac{1}{2}e^4 + \tfrac{1}{4}e^6 + \tfrac{5}{32}e^8 + \tfrac{7}{64}e^{10} + \cdots] \\ &\quad + \sqrt{(1+\nu)\,a}\,[1 - e^2 + \tfrac{1}{4}e^4 - \tfrac{1}{8}e^6 + \tfrac{1}{64}e^8 - \tfrac{3}{128}e^{10} + \cdots] + R. \end{aligned}$$

This equation contains as variables only e and θ; hence, since e should never be negative, the dependence of the two variables on each other may be shown graphically by taking e as the radius and θ as the angle in a system of polar coördinates. If we are given a pair of simultaneous values of e and θ, it is obvious that by their aid we can determine the constant value of W. Desiring to ascertain at what points on the axis the curve passes we make in (61) in succession $\theta = 0°$ and $\theta = 180°$ and we get two equations of the forms:

$$(62) \quad \begin{cases} D = \quad M_1 e + M_2 e^2 + M_3 e^3 + M_4 e^4 + M_5 e^5 + \cdots, \\ D = -M_1 e + M_2 e^2 - M_3 e^3 + M_4 e^4 - M_5 e^5 + \cdots, \end{cases}$$

where D may be regarded as the arbitrary constant and the M are constants, being functions of a and ν. These equations are transcendental in e and are such that the positive roots of the one are equivalent to the negative roots of the

other. If each has a positive real root continuous with the value of e which was used for the determination of the constant value of W, the variable θ generally moves through the whole range of real values. But, if the first equation has two positive real roots and the second none, θ will librate about the value $\theta = 0°$. But, if the second has two positive real roots and the first none, θ will librate about $\theta = 180°$. It will be seen that when $D = 0$ we have the limit separating non-libration from libration.

XI.

Case I.—*Non-libration.*—Here, as θ goes through the semicircumference, it can be employed as the independent variable. Then, in the first equation of (60), we assign to θ, in succession, a series of equidistant values covering the semicircumference. (Those used in our illustrative examples are 13 in number, viz, $\theta = 0°$, $\theta = 15°$, $\theta = 30°$, $\cdots$, $\theta = 180°$.) This procedure furnishes us with a like number of equations for determining the corresponding values of e. Solving these by the tentative process we have these values of e, and can apply to them the procedure of mechanical quadratures. Thus is obtained a general expression for e as a periodic function of θ involving only cosines.

As the next step these special values of e can be substituted in the right member of the second equation of (60). To the special values thus obtained for $d\tau/d\theta$ can be applied mechanical quadratures, and the resulting periodic series, involving only cosines of integral multiples of θ, can be integrated with respect to this variable. This integral may be put in the form:

$$\theta_0(t + c) = \theta + \beta_1 \sin\theta + \beta_2 \sin 2\theta + \beta_3 \sin 3\theta + \cdots. \tag{63}$$

Knowing θ_0 we are now in possession of the period of the inequalities we are endeavoring to derive. The left member of this equation we shall designate as the time-argument, and, for brevity, denote it as ζ. In the next place we assign to ζ a series of equidistant values going from 0° to 180°, and, by a tentative process applied to (63), arrive at the corresponding values of θ. These corresponding values of θ can be substituted for θ in the expression of e as a periodic function of θ, and thus we shall have the values of e which correspond to the equidistant values of ζ. We can now readily derive the similar values of the two quantities $e\cos\theta$ and $e\sin\theta$. To these we apply mechanical quadratures and thus obtain the periodic developments of these quantities in terms of ζ.

As the last step in this work we can, through the last equation of (60), express $dg/d\tau$ as a function of e and θ, and, by the substitution of the special values of the latter variables, obtain the special values of $dg/d\tau$ which correspond to the equidistant values of ζ. To these apply mechanical quadratures and the periodic series for $dg/d\tau$ is obtained. This being integrated we have the series for g, and the solution of the problem is completed.

XII.

Case II.—*Libration.*—Here we are shut off from the use of θ as an independent variable on account of its not going through the semicircumference. But this difficulty is surmounted by substituting for it another variable which does move continuously from $-\infty$ to $+\infty$. In order to ascertain, in the case of libration, the limiting values of θ we have to solve the simultaneous equations:

$$W = a \text{ constant}, \qquad \frac{\partial W}{\partial e} = 0,$$

the unknowns being e and θ. That is to say, a value of θ must be found which will make the first equation have two equal roots for e. This can be done by a tentative process. If we assume θ too large, generally, we shall not be able to discover real values for e from the first equation; but, if θ is taken too small, we get two values real but unequal for e. These two conditions must be brought as close as possible until we discover the point of passage from one to the other. In our illustrative example we escape the necessity of this tentative process by assuming as one of the two fundamental elements of the example not the D of (62) but the amount of libration.

The amount of libration being thus either assumed or determined, let κ denote the limiting value of $\sin\theta$; we then can put

$$\sin\theta = \kappa \sin\psi; \tag{64}$$

and the motion of ψ can be regarded as extending continuously from $-\infty$ to $+\infty$. Adopting the variable ψ for replacing θ, the second equation of (60) takes the form:

$$\frac{d\tau}{d\psi} = \sqrt{(1+\nu)a}\,\frac{e}{\sqrt{1-e^2}\left[2-\sqrt{1-e^2}\right]^2}\,\frac{\kappa\cos\psi}{\sqrt{1-\kappa^2\sin^2\psi}}\,\frac{1}{\dfrac{\partial W}{\partial e}}, \tag{65}$$

where the newly introduced radical must receive the sign of $\cos\theta$. We can now make ψ play the same rôle as θ did in Case I, and there is need of no further explanations.

XIII.

We attend now to the integration of equations (60). *The operation of Delaunay's lunar theory which is numbered 23* has great affinity with that here detailed, and the two may be compared.* He, it is true, has six variables to our four; but, in comparing, his γ should be made to vanish and his h then becomes indeterminate.

* Mémoires de l'Académie des Sciences, vol. XXVIII, p. 493.

The periodic development of the reciprocal of the distance between two planets as a function of the time has been given by LEVERRIER to terms of the seventh order inclusive, and those of the eighth order have afterwards been added by M. BOUQUET.* We avail ourselves of this development and adopt the mode of LEVERRIER for noting the coefficients except in the portion which is a function of e alone. We put $A_j^{(i)} = (1/j!)\, a^j\, d^j b_{\frac{1}{2}}^{(i)}/da^j$, $j = 0$ in the portion factored by $\cos 0\theta$, $j = 2$ in the portion factored by $\cos\theta$, $j = 4$ in the portion factored by $\cos 2\theta$ and so on; only the numerical factors are written since the A can easily be filled in as they always commence with $A_0^{(i)}$, and the lower index always increases by a unit in each step to the right. With LEVERRIER we put χ for $\frac{1}{2}e$. This then is the development of a'/Δ, preserving only the terms involving the integral multiples of θ as arguments:

$$
\begin{aligned}
\frac{a'}{\Delta} = {} & \tfrac{1}{2}A_0^{(0)} + [A_1^{(0)} + A_2^{(0)}]\chi^2 + \tfrac{3}{1}[A_3^{(0)} + A_4^{(0)}]\chi^4 + \tfrac{4.5}{1.2}[A_5^{(0)} + A_6^{(0)}]\chi^6 + \tfrac{5.6.7}{1.2.3}[A_7^{(0)} + A_8^{(0)}]\chi^8 \\
& + \Big\{ -[4+1]\chi + \Big[14 + \tfrac{5}{2} - 6 - 3\Big]\chi^3 + \Big[-\tfrac{5}{3} - \tfrac{53}{12} + \tfrac{34}{3} + 5 - 16 - 10\Big]\chi^5 \\
& \qquad + \Big[\tfrac{271}{36} - \tfrac{203}{144} + \tfrac{19}{4} - \tfrac{49}{8} + 20 + \tfrac{25}{2} - 50 - 35\Big]\chi^7 \Big\} \cos\theta \\
& + \Big\{ \quad [22+7+1]\chi^2 + \Big[-\tfrac{596}{3} - \tfrac{212}{3} + 8 + 16 + 4\Big]\chi^4 + \Big[\tfrac{1300}{3} + \tfrac{743}{3} - 49 \\
& \qquad - 112 + 2 + 45 + 15\Big]\chi^6 + \Big[-\tfrac{8312}{45} - \tfrac{500212}{45} - \tfrac{280}{9} + \tfrac{1228}{5} \\
& \qquad\qquad + 0 - 236 - 16 + 140 + 56\Big]\chi^8 \Big\} \cos 2\theta \\
& + \Big\{ -\Big[134 + \tfrac{93}{2} + 10 + 1\Big]\chi^3 + \Big[\tfrac{4053}{2} + \tfrac{6289}{8} + 107 - \tfrac{117}{2} - 32 - 5\Big]\chi^5 \\
& \qquad + \Big[-\tfrac{64177}{40} - \tfrac{384789}{80} - \tfrac{12123}{20} + \tfrac{6249}{8} + 348 - \tfrac{189}{2} - 102 - 21\Big]\chi^7 \Big\} \cos 3\theta \\
(66) \qquad & + \Big\{ \quad \Big[\tfrac{2570}{3} + \tfrac{932}{3} + 80 + 13 + 1\Big]\chi^4 + \Big[-\tfrac{275528}{15} - \tfrac{109972}{15} - 1676 + 48 + 176 \\
& \qquad + 54 + 6\Big]\chi^6 + \Big[\tfrac{6259444}{45} + \tfrac{592976}{9} + \tfrac{234338}{15} - \tfrac{44756}{15} - \tfrac{10036}{3} - \tfrac{1432}{3} \\
& \qquad\qquad + 400 + 196 + 28\Big]\chi^8 \Big\} \cos 4\theta \\
& + \Big\{ -\Big[\tfrac{33797}{6} + \tfrac{50345}{24} + \tfrac{1795}{3} + \tfrac{245}{2} + 16 + 1\Big]\chi^5 + \Big[\tfrac{5652235}{36} + \tfrac{9141589}{144} + \tfrac{210217}{12} \\
& \qquad + \tfrac{48985}{24} - \tfrac{2020}{3} - \tfrac{775}{2} - 82 - 7\Big]\chi^7 \Big\} \cos 5\theta \\
& + \Big\{ \quad \Big[\tfrac{188616}{5} + \tfrac{71499}{5} + 4357 + 1024 + 174 + 19 + 1\Big]\chi^6 + \Big[-\tfrac{45378432}{35} \\
& \qquad - \tfrac{3695460}{7} - \tfrac{803616}{5} - \tfrac{150828}{5} + 80 + 2156 + 720 + 116 + 8\Big]\chi^8 \Big\} \cos 6\theta
\end{aligned}
$$

*Annales de l'Observatoire de Paris, vols. I, XIX.

$$+\left\{-\frac{46064791}{2520}+\frac{70738549}{720}+\frac{1880921}{60}+\frac{193921}{24}+\frac{4844}{3}+\frac{469}{2}-22-1\right\}\chi^7\cos 7\theta$$

$$+\left\{\frac{552146674}{315}+\frac{213998824}{315}+\frac{2018552}{9}+\frac{926516}{15}+\frac{41380}{3}+\frac{7192}{3}\right.$$

$$\left.+304+25+1\right\}\chi^8\cos 8\theta.$$

From this expression we must eliminate the variable a by means of its value in terms of e given by (57). We put

$$p=[2-\sqrt{1-e^2}]^{-2}-1.$$

Let LEVERRIER'S coefficient of $\cos j\theta$ be denoted thus:

$$c_0A_0^{(2j)}+c_1A_1^{(2j)}+c_2A_2^{(2j)}+\cdots=\Sigma\, c_iA_i^{(2j)},$$

where c_i is a function of e. Denoting the similar coefficient, after the variable a has been eliminated through (57) by $\Sigma f_iA_i^{(2j)}$, we evidently have

$$f_i=(1+p)^ic_i+\frac{i}{1}(1+p)^{i-1}pc_{i-1}+\frac{i(i-1)}{1.2}(1+p)^{i-2}p^2c_{i-2}+\cdots.$$

By means of this formula we obtain the following expression, in which a, the argument of the various quantities $A_i^{(2j)}$, is the constant a of (57):

$$\frac{a'}{\Delta}=\frac{1}{2}+[0-1+1]\chi^2+[0+0-8-9+3]\chi^4+[0+0+48+100+0-50+10]\chi^6$$

$$+[0-4-160-600-224+720+240-245+35]\chi^8$$

$$+\left\{-[4+1]\chi+\left[14+\frac{45}{2}+2-3\right]\chi^3+\left[-\frac{5}{3}-\frac{1325}{12}-\frac{218}{3}+65+32-10\right]\chi^5\right.$$

$$\left.+\left[\frac{271}{36}+\frac{33829}{144}+\frac{6209}{12}-\frac{3025}{8}-796+\frac{105}{2}+190-35\right]\chi^7\right\}\cos\theta$$

$$+\left\{[22+7+1]\chi^2+\left[-\frac{596}{3}-\frac{560}{3}-56+4+4\right]\chi^4+\left[\frac{1300}{3}+1557+\frac{3517}{3}\right.\right.$$

$$\left.+248-222-35+15\right]\chi^6+\left[-\frac{8312}{45}-\frac{740632}{45}-\frac{95032}{9}-\frac{19112}{5}\right.$$

$$\left.\left.+3680+2184-496-280+56\right]\chi^8\right\}\cos 2\theta$$

$$+\left\{-\left[134+\frac{93}{2}+10+1\right]\chi^3+\left[\frac{4053}{2}+\frac{12065}{8}+559+\frac{147}{2}-16-5\right]\chi^5\right.$$

$$\left.+\left[-\frac{64177}{40}-\frac{1400349}{80}-\frac{248943}{20}-\frac{26439}{8}+612+\frac{971}{2}+18-21\right]\chi^7\right\}\cos 3\theta$$

(67)

$$+\left\{\left[\frac{2570}{3}+\frac{932}{3}+80+13+1\right]\chi^4+\left[-\frac{275528}{15}-\frac{60004}{5}-\frac{14404}{3}-1068-48\right.\right.$$

$$\left.+34+6\right]\chi^6+\left[\frac{6259444}{45}+\frac{1602224}{9}+\frac{1180994}{15}+\frac{630004}{15}\right.$$

$$\left.\left.+\frac{11372}{3}-\frac{7912}{2}-800+28+28\right]\chi^8\right\}\cos 4\theta$$

$$+\left\{-\left[\frac{33797}{6}+\frac{50345}{24}+\frac{1795}{3}+\frac{245}{2}+16+1\right]\chi^5+\left[\frac{5652235}{36}+\frac{13594381}{144}\right.\right.$$
$$\left.\left.+\frac{469037}{12}+\frac{256585}{24}+\frac{4628}{3}-\frac{95}{2}-58-7\right]\chi^7\right\}\cos 5\theta$$
$$+\left\{\left[\frac{188616}{5}+\frac{71499}{5}+4357+1024+174+19+1\right]\chi^6+\left[-\frac{45378432}{35}\right.\right.$$
$$\left.\left.-\frac{5152104}{7}-\frac{1549888}{5}-\frac{473688}{5}-19088-1704+240+88+8\right]\chi^8\right\}\cos 6\theta$$
$$+\left\{-\frac{46064791}{2520}+\frac{70738549}{720}+\frac{1880921}{60}+\frac{191863}{24}+\frac{4844}{3}+\frac{469}{2}+21+1\right\}\chi^7\cos 7\theta$$
$$+\left\{\frac{552146674}{315}+\frac{213998824}{315}+\frac{2018552}{9}+\frac{926516}{15}+\frac{41380}{3}+\frac{7192}{3}+304+25+1\right\}\chi^8\cos 8\theta\,.$$

In forming the value of $dg/d\tau$ we need to know the derivative of the foregoing expression with respect to a. By noting the equation:

$$a\frac{\partial A_i^{(2j)}}{\partial a}=iA_i^{(2j)}+(i+1)A_{i+1}^{(2j)}\,,$$

and changing our mode of noting the coefficients so that the number first given is the coefficient of $A_1^{(2j)}$ instead of $A_0^{(2j)}$, we have:

$$a\frac{\partial\frac{a'}{\Delta}}{\partial a}=\tfrac{1}{2}+[-1+0+3]\chi^2+[0-16-51-24+15]\chi^4+[0+96+444+400-250$$
$$-240+70]\chi^6+[-4-328-2280-3296+2480+5760-35-1680+315]\chi^8$$
$$+\left\{-[5+2]\chi+\left[\frac{73}{2}+49-3-12\right]\chi^3+\left[-\frac{1345}{12}-\frac{2197}{6}-23+388+110-60\right]\chi^5\right.$$
$$\left.+\left[\frac{34913}{144}+\frac{108337}{72}+\frac{3343}{8}-\frac{9093}{2}-\frac{7435}{2}+1455+1085-280\right]\chi^7\right\}\cos\theta$$
$$+\left\{[29+16+3]\chi^2+\left[-\frac{1156}{3}-\frac{1456}{3}-156+32+20\right]\chi^4+\left[\frac{5971}{3}\right.\right.$$
$$\left.+\frac{16376}{3}+4261+104-1285-120+105\right]\chi^6+\left[-\frac{83216}{5}-\frac{270176}{5}\right.$$
$$\left.\left.-\frac{647168}{15}-\frac{2848}{5}+29320+10128-5432-1792+504\right]\chi^8\right\}\cos 2\theta$$
$$+\left\{-\left[\frac{361}{2}+113+33+4\right]\chi^3+\left[\frac{28277}{8}+\frac{16537}{4}+\frac{3795}{2}+230-105-30\right]\chi^5\right.$$
$$+\left[-\frac{1528703}{80}-\frac{2396121}{40}-\frac{1890243}{40}-\frac{21543}{2}+\frac{11975}{2}\right.$$
$$\left.\left.+3021-21-168\right]\chi^7\right\}\cos 3\theta$$
$$+\left\{\left[\frac{3502}{3}+\frac{2344}{3}+279+56+5\right]\chi^4+\left[-\frac{91108}{3}-\frac{504064}{15}-17608-4464\right.\right.$$
$$\left.\left.-70+240+42\right]\chi^6+\left[\frac{14270564}{45}+\frac{23108204}{45}+\frac{1810998}{5}+\frac{2747456}{15}\right.\right.$$
$$\left.\left.+\frac{17300}{5}-20624-5404+448+252\right]\chi^8\right\}\cos 4\theta \qquad (68)$$

$$+\left\{-\left[\frac{185533}{24}+\frac{64705}{12}+\frac{4325}{2}+554+85+6\right]\chi^5+\left[\frac{36203321}{144}+\frac{19222825}{72}\right.\right.$$
$$\left.\left.+\frac{1194659}{8}+\frac{293609}{6}+\frac{44855}{6}-633-455-56\right]\chi^7\right\}\cos 5\theta$$
$$+\left\{\left[52023+\frac{186568}{5}+16143+4792+965+120+7\right]\chi^6\right.$$
$$+\left[-\frac{71138952}{35}-\frac{73219472}{35}-\frac{6070728}{5}-\frac{2276512}{5}-103960-8784\right.$$
$$\left.\left.+2296+768+72\right]\chi^8\right\}\cos 6\theta$$
$$-\left\{\frac{587299425}{5040}+\frac{93309601}{360}+\frac{4721157}{40}+\frac{230615}{6}+\frac{55475}{6}+1539+161+8\right\}\chi^7\cos 7\theta$$
$$+\left\{\frac{766145498}{315}+\frac{569296288}{315}+\frac{12872308}{15}+\frac{4533664}{15}+\frac{242860}{3}+16208\right.$$
$$\left.+2303+208+9\right\}\chi^8\cos 8\theta.$$

It is desirable to have the means of verifying these truncated developments of a'/Δ derived from the work of LEVERRIER and M. BOUQUET. In fact, by the application of the first of two following theorems, an error has been found in M. BOUQUET'S expression for (225); in the coefficient of K_3, $-h$ should be substituted for h. The two theorems are the following:

The coefficient of $\cos j\theta$ in the periodic development of a'/Δ is the same as that of s^j in the expansion of the expression

$$\sum_{i=0}^{i=\infty} A_i^{(2j)}(-\chi)^i\left(s+\frac{1}{s}\right)^i\left[1-\chi\left(s+\frac{1}{s}\right)\right]\left(\frac{s-\omega}{1-\omega s}\right)^{2j} e^{j\lambda\left(s-\frac{1}{s}\right)}$$

in a power series with reference to s.

The coefficient of $\cos j\theta$ in the same development after a has been replaced by $a[2-\sqrt{1-e^2}]^2$ is the same as that of s^j in the expansion of the expression

$$\sum_{i=0}^{i=\infty} A_i^{(2j)}\left[p-\chi(1+p)\left(s+\frac{1}{s}\right)\right]^i\left[1-\chi\left(s+\frac{1}{s}\right)\right]\left(\frac{s-\omega}{1-\omega s}\right)^{2j} e^{j\lambda\left(s-\frac{1}{s}\right)}$$

in a power series with reference to s.

In these expressions ω stands for $e/(1+\sqrt{1-e^2})$.

XIV.

The two linear elements which determine all the coefficients in the periodic developments involved in this problem may be taken to be the constant a of (57) and the constant D of (62). It is proposed to elaborate two examples illustrating the subject in hand, one exhibiting non-libration, the other libration. In both we will assign to a such a value as makes $\log a = 9.8$. This value makes the period of revolution of the small planet nearly or exactly half that of

Jupiter. Whether we are to have a case of non-libration or libration will then depend on the value assigned to the second constant D.

In the first place then we compute the values of such of the quantities $A_i^{(j)}$ as are needed in this investigation, correspondent to $\log a = 9.8$, by procedures which it is unnecessary to detail. The results are contained in the following table:

Values of $\log A_i^{(j)}$ *for* $\log a = 9.8$.

	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
0	0.354 4041 774	9.845 4797 897	9.935 0116 655	9.989 1230	0.111 3716
2	9.564 3962 993	9.965 8367 1	0.002 7463 5	0.007 7852	0.121 2342
4	9.035 0709 047	9.694 9897 1	9.992 6030 6	0.098 8192	0.164 1565
6	8.555 0516 205	9.374 1611 5	9.842 7442 0	0.099 5969	0.231 3986
8	8.096 8549 86	9.031 7132 5	9.624 7440 8	0.005 5654	0.239 5776
10	7.651 0634 5	8.677 0326 9	9.367 4385 2	9.849 2650	0.178 8047
12	7.213 2942 8	8.314 4461 7	9.084 6837 2	9.649 5418	0.062 8948
14	6.781 1495	7.946 3186 8	8.784 1408 8	9.419 6593	9.905 2677
16	6.353 1544	7.574 0873	8.470 4851	9.167 1668	9.715 8220

j	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
0	0.251 2555	0.408 5603	0.576 748	0.753 37	0.937 0
2	0.257 4382	0.412 7086	0.579 698	0.755 57	0.938 5
4	0.278 2372	0.426 0622	0.588 970	0.762 42	0.943 9
6	0.329 4253	0.453 2441	0.606 143	0.774 66	0.956 8
8	0.385 1308	0.503 3035	0.637 049	0.794 30	0.977 1
10	0.398 9502	0.552 5313	0.684 255	0.826 57	1.004 5
12	0.360 4400	0.570 3163	0.729 206	0.870 86	1.029 0
14	0.272 3635	0.545 6897	0.750 571	0.909 38	1.045 5
16	0.146 2268	0.480 0088	0.721 959	0.935 78	1.045 7

Substituting these values in (67) we get

$$
\begin{aligned}
\frac{a'}{\Delta} = {} & 1.1307697497 + 0.04010033e^2 - 0.7367846e^4 + 1.17661e^6 + 0.6155e^8 \\
& + [-1.19571949 + 3.1113902e^2 - 2.669146e^4 - 1.90033e^6]e\cos\theta \\
& + [1.70905245 - 9.8883917e^2 + 30.18579e^4 - 62.2057e^6]e^2\cos 2\theta \\
& + [-3.00445698 + 27.190861e^2 - 117.01214e^4]e^3\cos 3\theta \\
& + [5.7966694 - 71.99282e^2 + 369.2943e^4]e^4\cos 4\theta \\
& + [-11.800399 + 186.12652e^2]e^5\cos 5\theta \\
& + [24.86635 - 475.7506e^2]e^6\cos 6\theta \\
& - 52.40299e^7\cos 7\theta \\
& + 118.0918e^8\cos 8\theta .
\end{aligned} \tag{69}
$$

We adopt the mass of Jupiter so that $\nu = 1/1047.355$. Then, in the expression (61) of W, the portion which is independent of the interaction of Jupiter and the small planet, developed in powers of e^2, becomes:

$$
\begin{aligned}
& 1.58715\ 39467\ 862 \\
& -0.00226\ 07543\ 2\ e^2 \\
& +0.59490\ 01358\ e^4 \\
& +0.09877\ 323\ e^6 \\
& +0.13623\ 71\ e^8 \\
& +0.06804\ 8\ e^{10}.
\end{aligned}
$$

If we omit from the expansion of W its constant term, and call D the constant of the thus modified W, as in (62), we have, as an integral of our problem,

$$
\begin{aligned}
D = & -0.00222246709e^2 + 0.5941966641e^4 + 0.09989664\ e^6 + 0.1368248e^8 \\
& + [-0.00114165636 + 0.0029707121e^2 - 0.002548463e^4 - 0.00181441e^6]e\cos\theta \\
& + [0.00163177054 - 0.0094412989e^2 + 0.02882097\ e^4 - 0.593931e^6]e^2\cos 2\theta \\
& + [-0.0028686138 + 0.025961456\ e^2 - 0.11172156\ e^4]e^3\cos 3\theta \\
& + [0.0055345794 - 0.06873774\ e^2 + 0\ 3525971\ e^4]e^4\cos 4\theta \\
& + [-0.01126686 + 0.1777110e^2]\ e^5\cos 5\theta \\
& + [0.02374204 - 0.4542400e^2]\ e^6\cos 6\theta \\
& - 0.0500337e^7\cos 7\theta \\
& + 0.1127524e^8\cos 8\theta .
\end{aligned} \tag{70}
$$

By making $\theta = 0°$ in the preceding equation, we get, as the correspondent of the first equation of (62), the following:

$$
\left.\begin{matrix}
-0.00114\ 16563\ 6\ e - 0.00059\ 06875\ 5\ e^2 \\
+0.00010\ 20983\ 5\ e^3 + 0.59028\ 99445\ \ e^4 \\
+0.01214\ 61357\ \ e^5 + 0.08372\ 1913\ \ e^6 \\
+0.01414\ 140\ \ e^7 + 0.08854\ 10\ \ e^8
\end{matrix}\right\} = D . \tag{71}
$$

It will be seen by comparison of the coefficients of this equation that, unless e is very small, it will not do to regard the equation as approximately a quadratic in e; for $e = 0.1$ the term in e^4 is ten times more important than the term in e^2. The supposition that the mean motion of the small planet is nearly double that of Jupiter makes the coefficient of e^2 nearly vanish. In fact a very small change in the adopted value of a would make this coefficient 0.

What sort of a curve we shall have exhibiting graphically the connection between e and θ will depend on the value assigned to D. To bring this out in a clear manner we compute the values of the left member of the preceding equation for each 0.01 in the value of e between the limits ± 0.3, and thus have the following table:

e	D	e	D	e	D	e	D
−0.30	+0.0051 0216	−0.15	+0.0004 5648	+0.01	−0.0000 1148	+0.16	+0.0001 9224
0.29	44 8080	0.14	3 7472	0.02	2298	0.17	2 8623
0.28	39 2024	0.13	3 0675	0.03	3130	0.18	4 0095
0.27	34 1625	0.12	2 5066	0.04	4509	0.19	5 3896
0.26	29 6473	0.11	2 0477	0.05	5485	0.20	7 0298
0.25	25 6178	0.10	1 6715	0.06	6294	0.21	8 9590
0.24	22 0365	0.09	1 3659	0.07	6857	0.22	11 2071
0.23	18 8675	0.08	1 1166	0.08	7082	0.23	13 8067
0.22	16 0764	0.07	9115	0.09	6861	0.24	16 7912
0.21	13 6306	0.06	7400	0.10	6073	0.25	20 1960
0.20	11 4987	0.05	5928	0.11	4582	0.26	24 0580
0.19	9 6512	0.04	4622	0.12	−0.0000 2236	0.27	28 4159
0.18	8 0599	0.03	3420	0.13	+0.0000 1129	0.28	33 3102
0.17	6 6982	0.02	2269	0.14	5695	0.29	38 7834
−0.16	+0.0005 5411	−0.01	+0.0000 1136	+0.15	+0.0001 1657	+0.30	+0.0044 8802

As e ought always to be positive, in the first half of this table we may change the sign of e, provided we suppose that the corresponding value of D is regarded as appertaining to the special value 180° for θ, while, in the remainder of the table, this value corresponds to $\theta = 0°$.

From the course of the values of D in the table we see there is one minimum $= -0.00007\ 082$, which occurs for $e = 0.08$ about; consequently, if D is chosen greater than this the equation (71) will have two real roots for e. If D is positive one of these roots will be negative; changing the sign of the latter it will belong to the value $\theta = 180°$; the positive root will belong to $\theta = 0°$. Thus, in this case, the motion of θ is generally through the whole semicircumference, and hence is continuous from $-\infty$ to $+\infty$. But, if D is negative, both roots will be positive, and thus belong to $\theta = 0°$. In this case, therefore, θ departs from 0° and comes back to it without having reached 180°. This is called a libration; we see that $D = 0$ marks the dividing point between continuous and libratory motion for θ. The latter case also has the largest swing in the values of e, viz, from $e = 0$ to about $e = 0.127$. Generally, the larger D is, the smaller will be the variation in e. Thus, if $D = +0.0045$, e will vary from 0.29 to 0.30. If there is libration e cannot exceed 0.127. These remarks, however, must be understood as applying only to the values holding for $\theta = 0°$ and $\theta = 180°$. Larger values for e may obtain for values of θ lying between 0° and 180°.

XV.

For our illustrative example, in the case of a continuous motion for θ, we assign to D the value $+0.0001$ in (70). All the coefficients of the various periodic series will now have determinate numerical values. The preceding table shows that, for this assumption, the eccentricity will have, when $\theta = 0°$, the approximate value $e = 0.1475$, and, when $\theta = 180°$, the approximate value -0.0745. In this case these are the limiting values, as e continuously diminishes while θ is passing from 0° to 180°.

Attending now to the elaboration of our selected example, in (70) we give to θ, in succession, the values 15°, 30°, 45°, 60°, 75°, 90° and get as the right member of (70), [the value $\theta = 0°$ has already been considered in (70)],

θ				
15°	$-0.00110\ 27554e$	$-0.00080\ 93046e^2$	$+0.00084\ 1072e^3$	$+0.58878\ 755e^4$
30	98 87034	140 65774	257 2712	58670 872
45	80 72730	222 24671	412 9027	58866 208
60	57 08282	303 83568	435 3970	59615 002
75	$-0.00029\ 54824$	363 56296	$+0.00279\ 7293$	60514 036
90	0	$-0.00385\ 42466$	0	$+0.60917\ 254$
θ				
15°	$+0.01297\ 98e^5$	$+0.09048\ 7e^6$	$-0.02180\ 7e^7$	$+0.20531e^8$
30	$+0\ 00755\ 03$	$+0.12493\ 4$	$-0.11214\ 3$	$+0.32869$
45	$-0.01219\ 27$	$+0.16863\ 4$	$-0.08332\ 4$	-0.10302
60	$-0.03286\ 91$	$+0.14359\ 7$	$+0.17465\ 3$	-0.52039
75	$-0.02990\ 01$	$+0.04056\ 8$	$+0.29851\ 4$	$+0.30818$
90	0	$-0.02140\ 4$	0	$+1.11581$

The coefficients in the second quadrant for θ are the same as in the first, but in reverse order, except, that for the odd powers of e the sign must be reversed.

Making the right members of these 13 equations equal to + 0.0001, we solve them with reference to e as the unknown, and substitute, in succession, the values thus obtained and the corresponding value of θ in the right member of the second equation of (60). The results obtained are the following:

θ	e	$d\tau/d\theta$
0°	0.14746 2372	18.321384
15	0.14702 7366	18.443969
30	0.14569 2650	18.827993
45	0.14335 8947	19.521595
60	0.13982 8670	20.600475
75	0.13480 9503	22.197678
90	0.12791 2165	24.508178
105	0 11869 0066	27.615097
120	0.10709 4867	30.992715
135	0.09444 8152	32.866681
150	0.08360 2031	31.947061
165	0.07679 7488	29.983003
180	0.07454 7767	29.112462

The mean of the numbers in the third column, attributing half weight to the first and last, is 25.101781; and this is the number of revolutions of Jupiter in the period of the inequalities we are investigating. If the sidereal revolution of Jupiter is put at 11.861980 Julian years, the latter period is 297.75681 such years.

From the special values given in this table we can derive the two periodic series representing them. Integrating the latter, and, for brevity, putting ζ for $\theta_0(t + c)$, we get the following expressions:

$$e = \left\{\begin{array}{rl} & 0.11918\ 891 \\ +0.03553\ 171 & \cos\ \theta \\ -0.00857\ 010 & \cos\ 2\theta \\ +0.00123\ 337 & \cos\ 3\theta \\ +0.00027\ 721 & \cos\ 4\theta \\ -0.00029\ 767 & \cos\ 5\theta \\ +0.00012\ 029 & \cos\ 6\theta \\ -0.00001\ 828 & \cos\ 7\theta \\ -0.00000\ 783 & \cos\ 8\theta \\ +0.00000\ 780 & \cos\ 9\theta \\ -0.00000\ 374 & \cos 10\theta \\ +0.00000\ 037 & \cos 11\theta \\ +0.00000\ 033 & \cos 12\theta \end{array}\right\}, \qquad \zeta = \theta + \left\{\begin{array}{rl} -60173''.40 & \sin\ \theta \\ -1643.74 & \sin\ 2\theta \\ +4612.77 & \sin\ 3\theta \\ -2132.15 & \sin\ 4\theta \\ +544.20 & \sin\ 5\theta \\ +6.13 & \sin\ 6\theta \\ -87.51 & \sin\ 7\theta \\ +52.96 & \sin\ 8\theta \\ -16.90 & \sin\ 9\theta \\ -0.02 & \sin 10\theta \\ +3.89 & \sin 11\theta \\ -1.98 & \sin 12\theta \end{array}\right\}.$$

The first of these is simply a transformation of the equation $W = D$ by which e is expressed in terms of θ. From the second, by attributing to ζ in succession the 13 values 0°, 15°, 30°, ···, 180°, using a tentative process, we can get the corresponding values of θ. Thence by substitution in former results, the corresponding values of the four quantities e, $e\cos\theta$, $e\sin\theta$ and $d\tau/d\theta$ can be obtained. The results follow, the first column containing the argument:

ζ	θ	e	$e\cos\theta$	$e\sin\theta$	$d\tau/d\theta$
0°	0° 0′ 0.″00	0.14746 237	+ 0.14746 237	0.00000 000	18.321384
15	20 27 58.05	0.14664 872	0.13739 211	+ 0.05127 624	18.551258
30	40 25 23.50	0.14418 648	0.10976 569	0.09349 457	19.274084
45	59 23 15.52	0.13999 969	0.07129 163	0.12048 825	20.547277
60	76 57 25.45	0.13402 600	+ 0.03024 713	0.13056 830	22.453723
75	92 51 50.00	0.12634 244	− 0.00631 251	0.12618 464	25.044275
90	107 3 42.77	0.11722 678	0.03439 485	0.11206 745	28.087856
105	119 48 15.87	0.10725 784	0.05331 151	0.09307 055	30.953203
120	131 35 27.90	0.09727 656	0.06457 314	0.07275 327	32.686384
135	143 3 46.20	0.08820 764	0.07050 393	0.05300 740	32.666482
150	154 48 43.11	0.08092 658	0.07323 176	0.03444 156	31.300251
165	167 9 46.91	0.07619 338	0.07428 902	+ 0.01692 847	29.759410
180	180 0 0.00	0.07454 777	− 0.07454 777	0.00000 000	29.112462

From the numbers in the fourth and fifth columns are derived the series:

$$e\cos\theta = \left\{\begin{array}{l} +0.00071\ 143 \\ +0.10563\ 221\cos\zeta \\ +0.03542\ 782\cos 2\zeta \\ +0.00539\ 467\cos 3\zeta \\ +0.00031\ 839\cos 4\zeta \\ -0.00002\ 420\cos 5\zeta \\ -0.00000\ 130\cos 6\zeta \\ +0.00000\ 245\cos 7\zeta \\ +0.00000\ 111\cos 8\zeta \\ +0.00000\ 026\cos 9\zeta \\ -0.00000\ 044\cos 10\zeta \\ -0.00000\ 032\cos 11\zeta \\ +0.00000\ 030\cos 12\zeta \end{array}\right\}, \qquad e\sin\theta = \left\{\begin{array}{l} +0.11737\ 247\sin\zeta \\ +0.03373\ 708\sin 2\zeta \\ +0.00528\ 997\sin 3\zeta \\ +0.00035\ 675\sin 4\zeta \\ -0.00001\ 633\sin 5\zeta \\ -0.00000\ 316\sin 6\zeta \\ -0.00000\ 095\sin 7\zeta \\ -0.00000\ 062\sin 8\zeta \\ +0.00000\ 041\sin 9\zeta \\ +0.00000\ 018\sin 10\zeta \\ -0.00000\ 004\sin 11\zeta \end{array}\right\}.$$

These forms for the integrals of our problem are to be preferred since they can also be used for the case of libration.

XVI.

To complete the solution the periodic series giving the position of the perihelion must be derived. Using logarithms instead of the actual coefficients, the first term of the right member of the third equation of (60) has the expression:

$$\begin{aligned} \frac{2}{\sqrt{(1+\nu)a}}\, a\frac{\partial R}{\partial a} = {} & [6.9251786] + [7.126034]e^2 - [8.00771]e^4 + [7.9475]e^6 + [8.7115]e^8 \\ & + \{ - [7.9015032] + [8.284709]e^2 + [7.66255]e^4 - [8.9051]e^6 \}\, e\cos\theta \\ & + \{ \ [8.3084076] - [9.068286]e^2 + [9.55239]e^4 - [9.7559]e^6 \}\, e^2\cos 2\theta \\ & + \{ - [8.707357] \ + [9.66685]\ e^2 - [0.2895]\ e^4 \}\, e^3\cos 3\theta \\ & + \{ \ [9.104403] \ - [0.20025]\ e^2 + [0.9097]\ e^4 \}\, e^4\cos 4\theta \\ & + \{ - [9.50120] \ + [0.7008]\ e^2 \}\, e^5\cos 5\theta \\ & + \{ \ [9.89773] \ + [1.1811]\ e^2 \}\, e^6\cos 6\theta \\ & - [0.2839]e^7\cos 7\theta \\ & + [0.6902]e^8\cos 8\theta . \end{aligned}$$

The remaining terms of this expression for $dg/d\tau$ can readily be derived from the values of e and $d\tau/d\theta$ correspondent to the argument ζ which have just been given. Calling these the second part of $dg/d\tau$, we have the following results:

ζ	First Part	Second Part	$dg/d\tau$
0°	+0.00004 9796	−0.00579 1193	−0.00574 1397
15	0.00005 9848	0.00572 9341	0.00566 9493
30	0.00012 1787	0.00554 3667	0.00542 1880
45	0.00023 7477	0.00509 7143	0.00485 9666
60	0.00039 1638	0.00420 4082	0.00381 2444
75	0.00059 5969	0.00270 5347	−0.00210 9378
90	0.00084 3915	−0.00024 2584	+0.00060 1331
105	0.00109 9821	+0.00326 3944	0.00436 3765
120	0.00131 9503	0.00773 7116	0.00905 6619
135	0.00146 5221	0 01284 0064	0.01430 5285
150	0.00153 5401	0.01789 1412	0.01942 6813
165	0.00156 2180	0.02178 6499	0.02334 8679
180	+0.00156 9151	+0.02328 0093	+0.02484 9244

The quantities in the last column furnish the periodic series for $dg/d\tau$. The absolute term shows that the mean motion of the perihelion of the minor planet is 0.00490 0079 times the mean motion of Jupiter. The integration of this series gives the expression for g. These two expressions follow; (g) is the arbitrary constant added to complete the integral, and, in the second term, the unit of t is a Julian year.

$$\frac{dg}{d\tau} = \left\{ \begin{array}{ll} +0\ 00490\ 0079 & \\ -0.01441\ 6898 & \cos\ \zeta \\ +0\ 00444\ 6030 & \cos\ 2\zeta \\ -0.00080\ 4772 & \cos\ 3\zeta \\ +0.00017\ 6760 & \cos\ 4\zeta \\ -0.00006\ 7470 & \cos\ 5\zeta \\ +0.00003\ 1972 & \cos\ 6\zeta \\ -0.00000\ 2614 & \cos\ 7\zeta \\ +0.00000\ 0139 & \cos\ 8\zeta \\ -0.00000\ 3981 & \cos\ 9\zeta \\ -0.00000\ 1706 & \cos 10\zeta \\ +0.00000\ 0414 & \cos 11\zeta \\ +0.00000\ 0648 & \cos 12\zeta \end{array} \right\}, \qquad g = \left\{ \begin{array}{ll} (g) + 535''.3662\ t & \\ -74645.14 & \sin\ \zeta \\ +11509\ 92 & \sin\ 2\zeta \\ -\ 1388.93 & \sin\ 3\zeta \\ +\ 228.80 & \sin\ 4\zeta \\ -\ 69.87 & \sin\ 5\zeta \\ +\ 27.59 & \sin\ 6\zeta \\ -\ 1.93 & \sin\ 7\zeta \\ +\ 0.09 & \sin\ 8\zeta \\ -\ 2.27 & \sin\ 9\zeta \\ -\ 0.88 & \sin 10\zeta \\ +\ 0.19 & \sin 11\zeta \\ +\ 0.28 & \sin 12\zeta \end{array} \right\}.$$

It is of interest to know the mean motion of the small planet which is not obvious at the beginning of the solution. We have the equation:

$$\frac{d(l+g)}{d\tau} = \frac{d\theta}{d\tau} - \frac{dg}{d\tau} + 2.$$

Substituting in the right member the mean motions of θ and g, its value is found to be 2.03493 7731; then, if for Jupiter we have $\mu' = 299''.12838$, for the small planet $\mu = 608''.70762$.

XVII.

Illustration in the Case of Libration.—In the example we have chosen to illustrate the theory, libration, when it exists, is always about the value $\theta = 0°$. In addition to the value $\log a = 9.8$ let us assume that the D of (70)

is to be so chosen that the half-swing of θ may be $50°$. Making, therefore, $\theta = 50°$ in (70), we get the first of the following equations in e, and the second by taking the derivative of the first with respect to e:

$$\left.\begin{array}{l} -0.00073\ 38426\ e \\ -0.00250\ 58226\ e^2 \\ +0.00439\ 3829\ e^3 \\ +0.58063\ 532\ e^4 \\ -0.02026\ 791\ e^5 \\ +0.17135\ 53\ e^6 \\ -0.01446\ 68\ e^7 \\ -0.32494\ 7\ e^8 \end{array}\right\} = D, \qquad \left.\begin{array}{l} -0.00073\ 38426 \\ -0.00501\ 16452\ e \\ +0.01318\ 1487\ e^2 \\ +2.36254\ 128\ e^3 \\ -0.10133\ 955\ e^4 \\ +1.02813\ 18\ e^5 \\ -0.10126\ 76\ e^6 \\ -2.59957\ 6\ e^7 \end{array}\right\} = 0.$$

Both of these equations should be satisfied when θ is at the limit of its swing, viz., when $\theta = \pm 50°$. The root of the second equation which is applicable to our purpose is $e + 0.07606\ 124$, and this value substituted in the first gives $D = -0.00048\ 63102$, which is the value of D which brings about a libration of $50°$.

From the equation $\sin\theta = \sin 50° \sin\psi$ we obtain the following corresponding values:

ψ	θ		
°	°	′	″
0	0	0	0.00
15	11	26	8.27
30	22	31	15.64
45	32	47	51.90
60	44	33	38.75
75	47	43	35.34
90	50	0	0.00

By means of these values we determine the form of (70) corresponding to the seven values of ψ. The coefficients are given in the following table (the small figures at the top of the columns denote the order of the final decimal):

ψ	e	e^2	e^3	e^4	e^5	e^6	e^7	e^8
0°	−114 16564[10]	− 59 0688[9]	+ 10 210[8]	+59028 994[8]	+1214 61[7]	+ 8372 2[6]	+ 1414[5]	+ 8854[5]
15	111 89923	71 5977	54 217	58936 012	1284 01	8713 9	− 640	15961
30	105 45925	106 9471	164 931	58751 743	1188 52	10353 7	6636	30349
45	95 96627	154 8255	291 587	58665 083	+ 487 32	13434 5	12360	30000
60	85 42473	202 7039	385 516	58769 012	− 671 35	16174 8	11285	+ 4731
75	76 79586	237 7533	429 468	58965 923	1663 22	17134 1	− 4928	−22703
90	− 73 38426	−250 5823	+439 383	+59063 532	−2026 79	+17135 5	+ 1447	−32495

The expressions in this table constitute the left members of 7 equations of the 8th degree; they must be equated to the same quantity $D = -0.00004\ 863102$. The two smallest real roots of each should be derived (they are those suited to our purpose). The connection of these roots with the variable ψ is settled in following way: the larger of the two roots is made to correspond to the value of ψ standing as the argument in the table, while the smaller is assigned to the value $180° - \psi$; the two roots being equal for $\psi = 90°$, the common value is

assigned to that value of ψ. This arrangement is made in order that ψ and τ may augment together. These values of e together with the corresponding values of θ are, in succession, substituted in (65); thus we have the values of $d\tau/d\psi$ corresponding to equidistant values of ψ. These results are contained in the following table:

ψ	e	$d\tau/d\psi$
0°	0.10846 187	37.23986
15	0.10765 795	37.70988
30	0.10518 818	39.14539
45	0.10088 191	41.57020
60	0.09452 189	44.76280
75	0.08605 349	47.65995
90	0.07606 124	48.00164
105	0.06601 592	44.40759
120	0.05744 757	38.68685
135	0.05102 653	33.34281
150	0.04671 683	29.47279
165	0.04426 767	27.22320
180	0.04347 566	26.49213

It should be noted that, in the computation of the third column, the factor $\cos\psi/(\partial W/\partial e)$ takes on the indeterminate form $0/0$; employing the usual method of treating vanishing fractions, this factor equals $-1/[(\partial^2 W/\partial e^2)(de/d\psi)]$. If here we should use the equation $W = D$ to determine $de/d\psi$, the result would again be indeterminate. But this difficulty is avoided by employing the value of e as a periodic function of ψ given by the quantities of the second column. Thus if

$$e = a_0 + a_1 \cos\psi + a_2 \cos 2\psi + a_3 \cos 3\psi + \cdots,$$

then

$$\frac{de}{d\psi} = -a_1 \sin\psi - 2a_2 \sin 2\psi - 3a_3 \sin 3\psi - \cdots,$$

and, for the special value $\psi = 90°$, this becomes:

$$\frac{de}{d\psi} = -a_1 + 3a_3 - 5a_5 + 7a_7 - \cdots.$$

For this special value of ψ it is found that

$$e\frac{\partial^2 W}{\partial e^2} = +0.00288\ 92836, \qquad \frac{de}{d\psi} = -0.03939\ 373.$$

The mean of the numbers in the last column of the table, allowing half weight to the first and last, is 38.65409, which is the number of revolutions of Jupiter contained in the period of libration; thus this period is 458.5144 Julian years. From the special values of e and $d\tau/d\psi$ given in the table we derive the periodic series representing them. The latter can be integrated, and, as before, we put ζ for $\theta_0(t + c)$. Thus we get the following expressions:

$$e = \left\{ \begin{array}{l} 0.0759\ 8399 \\ +0.0338\ 8957 \cos\ \psi \\ -0.0000\ 4144 \cos\ 2\psi \\ -0.0015\ 2988 \cos\ 3\psi \\ +0.0000\ 3031 \cos\ 4\psi \\ +0.0001\ 5016 \cos\ 5\psi \\ -0.0000\ 0467 \cos\ 6\psi \\ -0.0000\ 1929 \cos\ 7\psi \\ +0.0000\ 0062 \cos\ 8\psi \\ +0.0000\ 0281 \cos\ 9\psi \\ -0.0000\ 0012 \cos 10\psi \\ -0.0000\ 0026 \cos 11\psi \\ +0.0000\ 0008 \cos 12\psi \end{array} \right\}, \qquad \zeta = \psi + \left\{ \begin{array}{r} +29862.''37 \sin\ \psi \\ -20922.46 \sin\ 2\psi \\ -\ 416.31 \sin\ 3\psi \\ +\ 1650.00 \sin\ 4\psi \\ +\ 13.01 \sin\ 5\psi \\ -\ 193.31 \sin\ 6\psi \\ -\ 0.37 \sin\ 7\psi \\ +\ 27.39 \sin\ 8\psi \\ +\ 0.01 \sin\ 9\psi \\ -\ 4.65 \sin 10\psi \\ -\ 0.02 \sin 11\psi \\ +\ 0.81 \sin 12\psi \end{array} \right\}.$$

The first of these is simply a transformation of the equation $W = D$, by which e is expressed in terms of the auxiliary variable ψ.

Attributing to ζ, in succession, the 13 values $0°, 15°, 30°, \cdots, 180°$, by a tentative process we can get the corresponding values of ψ, as also, by substitution those of e, $e\cos\theta$ and $e\sin\theta$. These results follow:

ζ	ψ	e	$e\cos\theta$	$e\sin\theta$
0°	0° 0′ 0.″00	0.1084 6187	+0.1084 6187	0.0000 0000
15	15 30 0.64	0.1068 0283	0.1045 4084	+0.0218 6455
30	30 35 58.93	0.1022 4969	0.0941 5541	0.0398 7176
45	44 58 46.75	0.0956 9588	0.0804 5265	0.0518 1767
60	58 28 33.58	0.0879 7453	0.0666 2875	0.0574 4671
75	71 8 20.78	0.0797 1930	0.0549 1390	0.0577 8952
90	83 15 23.39	0.0714 4655	0.0463 7246	0.0543 5260
105	95 20 46.24	0.0636 2489	0.0411 4864	0.0485 2749
120	108 7 15.59	0.0566 9409	0.0388 6515	0.0412 7614
135	122 25 27.98	0.0510 7994	0.0389 6451	0.0330 2919
150	139 5 9.66	0.0468 5171	0.0405 2863	0.0235 0559
165	158 31 3.57	0.0443 2198	0.0425 4216	+0.0124 3393
180	180 0 0.00	0.0434 7566	+0.0434 7566	0.0000 0000

From the data of the fourth and fifth columns result the periodic series:

$$e\cos\theta = \left\{ \begin{array}{l} +0.0604\ 2349 \\ +0.0309\ 3374 \cos\ \zeta \\ +0.0147\ 2940 \cos\ 2\zeta \\ +0.0015\ 8319 \cos\ 3\zeta \\ +0.0007\ 3464 \cos\ 4\zeta \\ -0.0000\ 1013 \cos\ 5\zeta \\ +0.0000\ 6769 \cos\ 6\zeta \\ -0.0000\ 1229 \cos\ 7\zeta \\ +0.0000\ 1611 \cos\ 8\zeta \\ -0.0000\ 0669 \cos\ 9\zeta \\ +0.0000\ 0106 \cos 10\zeta \\ +0.0000\ 0528 \cos 11\zeta \\ -0.0000\ 0363 \cos 12\zeta \end{array} \right\}, \qquad e\sin\theta = \left\{ \begin{array}{l} +0.0571\ 8416 \sin\ \zeta \\ +0.0093\ 8541 \sin\ 2\zeta \\ +0.0030\ 1596 \sin\ 3\zeta \\ +0.0000\ 5256 \sin\ 4\zeta \\ +0.0001\ 9928 \sin\ 5\zeta \\ -0.0000\ 1597 \sin\ 6\zeta \\ +0.0000\ 1768 \sin\ 7\zeta \\ -0.0000\ 0390 \sin\ 8\zeta \\ +0.0000\ 0771 \sin\ 9\zeta \\ -0.0000\ 0714 \sin 10\zeta \\ +0.0000\ 0492 \sin 11\zeta \end{array} \right\}.$$

XVIII.

In computing, for this case, the values of $dg/d\tau$ by the third equation of (60) we make a like division into two parts as in the former case. Substituting the values of e and θ which correspond to the values $0°, 15°, 30°, \cdots, 180°$ of ζ we get the following special values:

ζ	First Part	Second Part	$dg/d\tau$
0°	+ 0.00019 583	− 0.05008 981	− 0.04989 398
15	20 725	4843 252	4822 527
30	24 656	4385 892	4361 236
45	30 013	3722 995	3692 982
60	36 434	2938 834	2902 400
75	42 685	2098 623	2055 938
90	47 979	1247 567	1199 588
105	51 817	− 0.00410 869	− 0.00359 052
120	54 013	+ 0.00397 237	+ 0.00451 250
135	54 781	1211 642	1266 423
150	54 497	1991 948	2046 445
165	53 746	2629 903	2683 649
180	+ 0.00053 360	+ 0.02891 583	+ 0.02944 943

From the quantities in the last column we obtain the periodic series for $dg/d\tau$, and thence by integration the expression for g; these results follow:

$$\frac{dg}{d\tau} = \left\{ \begin{array}{ll} -0.0116\ 3174 & \\ -0.0373\ 2443 & \cos\ \zeta \\ +0.0007\ 9929 & \cos\ 2\zeta \\ -0.0020\ 0735 & \cos\ 3\zeta \\ +0.0005\ 0972 & \cos\ 4\zeta \\ -0.0002\ 9143 & \cos\ 5\zeta \\ +0.0000\ 8500 & \cos\ 6\zeta \\ -0.0000\ 4117 & \cos\ 7\zeta \\ +0.0000\ 1080 & \cos\ 8\zeta \\ -0.0000\ 0438 & \cos\ 9\zeta \\ +0.0000\ 0251 & \cos 10\zeta \\ -0.0000\ 0293 & \cos 11\zeta \\ +0.0000\ 0214 & \cos 12\zeta \end{array} \right\}, \qquad g = \left\{ \begin{array}{ll} (g) - 1270''.844\ t & \\ -297586.8 & \sin\ \zeta \\ +\ 3186.4 & \sin\ 2\zeta \\ -\ 5334.9 & \sin\ 3\zeta \\ +\ 1016.0 & \sin\ 4\zeta \\ -\ 464.7 & \sin\ 5\zeta \\ +\ 113.0 & \sin\ 6\zeta \\ -\ 46.9 & \sin\ 7\zeta \\ +\ 10.8 & \sin\ 8\zeta \\ -\ 3.9 & \sin\ 9\zeta \\ +\ 2.0 & \sin 10\zeta \\ -\ 2.1 & \sin 11\zeta \\ +\ 1.4 & \sin 12\zeta \end{array} \right\}.$$

The unit of t in the second expression is a Julian year.

By using the same formula as in the former case we find that the mean μ of the small planet in this case has the value $609''.47474$.

XIX.

In attempting to apply the preceding method to the case where $D = 0$, we should find that $dg/d\tau$ became infinite at the point where $\theta = 0°$ or $\theta = 180°$, and, when D is quite small, we should have to deal with inconveniently large numbers. This difficulty is surmounted by computing the differentials of the two quantities $e\cos g$ and $e\sin g$ in place of that of g.

ON THE TYPES OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER IN THREE INDEPENDENT VARIABLES WHICH ARE UNALTERED BY THE TRANSFORMATIONS OF A CONTINUOUS GROUP*

BY

J. E. CAMPBELL

Introduction.

It is known (Messenger of Mathematics, November, 1898, p. 97) that the equation:

$$\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2} + \frac{\partial^2 z}{\partial x_3^2} = 0,$$

is unaltered by any infinitesimal transformation of a certain group of the tenth order, and that all such transformations which leave the equation unaltered are contained in the above group. It is at once evident that any equation which by a point transformation can be reduced to the above form will also be unaltered by the transformations of a group of the tenth order of like composition with the above. In the present paper a more general proposition is considered, viz., the form to which linear partial differential equations, of the second order in three independent variables, can be reduced which have the property of being unaltered for some infinitesimal transformations. Such equations form a class by themselves, the potential equation above and equations reducible to it by point transformation being only particular types of this class; it is here shown that the infinitesimal transformations which leave unaltered the equations of this class form in all cases a finite group of the eleventh order at highest; and certain types are tabulated to which all equations of the class may be reduced.

§1. *The definition equations of the group.*

Let x_1, x_2, x_3 be three independent variables, z the dependent variable, and let the linear partial differential equation of the second order be written:

$$\begin{aligned}(1)\qquad & a_{11}z_{11} + a_{22}z_{22} + a_{33}z_{31} + 2a_{23}z_{23} + 2a_{31}z_{31} + 2a_{12}z_{12} \\ & + 2a_1z_1 + 2a_2z_2 + 2a_3z_3 + az = 0;\end{aligned}$$

* Presented to the Society, December 28, 1899. Received for publication, December 29, 1899.

it is required to find the conditions that the equation may be unaltered by the infinitesimal transformation which consists in substituting for x_i, z,

$$x_i + \xi_i(x_1, x_2, x_3)t \quad (i=1,2,3),$$
$$z + z\zeta(x_1, x_2, x_3)t,$$

respectively, where t is a constant so small that its square may be neglected; in other words the conditions are required that the equation (1) may admit the infinitesimal transformation X where

$$X \equiv \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \xi_3 \frac{\partial}{\partial x_3} + z\zeta \frac{\partial}{\partial z}.$$

By a method almost identical with that given in the Proceedings of the London Mathematical Society, vol. 29, p. 251, we see that by this transformation z_i becomes $z_i + \pi_i t$ and $z_{i\kappa}$ becomes $z_{i\kappa} + \pi_{i\kappa} t$, where

$$\pi_i = \zeta z_i + z\frac{\partial \zeta}{\partial x_i} - \sum_{\kappa=1}^{\kappa=3} z_\kappa \frac{\partial \xi_\kappa}{\partial x_i},$$

$$\pi_{i\kappa} = z_{i\kappa}\zeta + z_i \frac{\partial \zeta}{\partial x_\kappa} + z_\kappa \frac{\partial \zeta}{\partial x_i} + z\frac{\partial^2 \zeta}{\partial x_i \partial x_\kappa} - \sum_{r=1}^{r=3} z_{\kappa r}\frac{\partial \xi_r}{\partial x_i} - \sum_{r=1}^{r=3} z_{ir}\frac{\partial \xi_r}{\partial x_\kappa} - \sum_{r=1}^{r=3} z_r \frac{\partial^2 \xi_r}{\partial x_i \partial x_\kappa}$$

$(i, \kappa = 1, 2, 3).$

The condition that (1) is unaltered implies that

$$\sum_{i,\kappa} a_{i\kappa}\pi_{i\kappa} + 2\sum_i a_i \pi_i + az\zeta$$
$$+ (\rho - \zeta)\left(\sum_{i,\kappa} a_{i\kappa} z_{i\kappa} + 2\sum_i a_i z_i + az\right)$$
$$+ \sum_{i,\kappa} (Xa_{i\kappa}) z_{i\kappa} + 2\sum_i (Xa_i) z_i + (Xa)z \equiv 0,$$

where $i, \kappa = 1, 2, 3$ and ρ is some undetermined function.

---

* LIE proves (Leipziger Berichte, 1894-95, p. 322) that a contact transformation which transforms every linear partial differential equation into a linear one of the same order must be of the form $x_1' = f_1(x_1, x_2, x_3)$, $x_2' = f_2(x_1, x_2, x_3)$, $x_3' = f_3(x_1, x_2, x_3)$, $z' = zf(x_1, x_2, x_3)$; and although it does not follow that some one particular linear equation may not have a more general transformation formula conserving its linearity, yet it may easily be verified by a method given in LIE-ENGEL (vol. II, p. 252 and §95) that no partial equation of the form $a_{11}z_{11} + a_{22}z_{22} + a_{33}z_{33}$ (and to such a form the equations here considered may be reduced) can conserve its linearity for an infinitesimal contact transformation unless the characteristic function w is of the form:

$$z_1 f_1(x_1, x_2, x_3) + z_2 f_2(x_1, x_2, x_3) + z_3 f_3(x_1, x_2, x_3) - zf(x_1, x_2, x_3) - \phi(x_1, x_2, x_3).$$

It follows that the infinitesimal transformations here considered are essentially of the most general form. See also Crelle's Journal, vol. 114, pp. 116-142; for this reference the author is indebted to Dr. C. L. BOUTON.

Substituting for $\pi_{i\kappa}$ and π_i and equating coefficients of z and its derivatives we get six equations of the form:

$$(2)\qquad a_{\kappa 1}\frac{\partial \xi_i}{\partial x_1}+a_{\kappa 2}\frac{\partial \xi_i}{\partial x_2}+a_{\kappa 3}\frac{\partial \xi_i}{\partial x_3}+a_{i1}\frac{\partial \xi_\kappa}{\partial x_1}+a_{i2}\frac{\partial \xi_\kappa}{\partial x_2}+a_{i3}\frac{\partial \xi_\kappa}{\partial x_3}=\rho a_{i\kappa}+Xa_{i\kappa},$$

these being obtained by equating the coefficients of the derivatives of the second order; and

$$(3)\qquad \left(\sum_{i,\kappa}a_{i\kappa}\frac{\partial^2}{\partial x_i\,\partial x_\kappa}+2\sum_i a_i\frac{\partial}{\partial x_i}\right)\xi_1 - 2\left(a_{11}\frac{\partial \zeta}{\partial x_1}+a_{12}\frac{\partial \zeta}{\partial x_2}+a_{13}\frac{\partial \zeta}{\partial x_3}\right)\equiv 2\rho a_1+2Xa_1,$$

and two other similar equations obtained from derivatives of the first order; and finally

$$(4)\qquad \left(\sum_{i,\kappa}a_{i\kappa}\frac{\partial^2}{\partial x_i\,\partial x_\kappa}+2\sum_i a_i\frac{\partial}{\partial x_i}\right)\zeta\equiv-\rho a-Xa,$$

obtained by equating the coefficients of z.

If values of ξ_1, ξ_2, ξ_3, ζ can be found to satisfy these ten equations then

$$\xi_1\frac{\partial}{\partial x_1}+\xi_2\frac{\partial}{\partial x_2}+\xi_3\frac{\partial}{\partial x_3}+z\zeta\frac{\partial}{\partial z}$$

is a transformation which the equation (1) admits. Now all such transformations obviously form a continuous group, since the operation equivalent to two such successive operations is also an operation which leaves the equation (1) unaltered; it is now to be proved that the group is a finite one.

§ 2. *Proof that the group is finite.*

In order to prove this the more readily it will be convenient to transform the equation (1) to one in which the coefficients of z_{23}, z_{31}, z_{12} shall be zero. First it should be stated that in this paper attention is confined to those equations of the form (1) for which the determinant,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

which we call M does not vanish. Now for any transformation of the type:

$$x_i'=f_i(x_1,\ x_2,\ x_3)\qquad (i=1,2,3),$$
$$z'=zf(x_1,\ x_2,\ x_3),$$

where f_1, f_2, f_3, f are any holomorphic functions, the equation (1) is transformed to another of the same type, the new coefficients being given by equations of the form:

$$a'_{12} f = \sum_{i,\kappa} a_{i\kappa} \frac{\partial x'_1}{\partial x_i} \frac{\partial x'_2}{\partial x_\kappa}.$$

It easily follows that

$$M' f^3 = M \left(\frac{\partial(x'_1, x'_2, x'_3)}{\partial(x_1, x_2, x_3)} \right)^2,$$

so that $M' \neq 0$ if $M \neq 0$.

Choosing x'_1, x'_2, x'_3 so that $a'_{23} = a'_{31} = a'_{12} = 0$, we see that the conditions (2), in order that the transformed (1) may admit

$$\xi'_1 \frac{\partial}{\partial x'_1} + \xi'_2 \frac{\partial}{\partial x'_2} + \xi'_3 \frac{\partial}{\partial x'_3} + \zeta' z' \frac{\partial}{\partial z},$$

take the simpler form (the accents being dropped):

$$(5) \qquad \begin{cases} 2a_{ii} \dfrac{\partial \xi_i}{\partial x_i} = \rho a_{ii} + X a_{ii}, \\ a_{\kappa\kappa} \dfrac{\partial \xi_i}{\partial x_\kappa} + a_{ii} \dfrac{\partial \xi_\kappa}{\partial x_i} = 0. \end{cases}$$

And since the determinant in the original equation does not vanish none of the quantities a_{11}, a_{22}, a_{33} can be zero.

We can now prove that the equation system (2), (3), (4) if consistent involves only a finite number of constants in its solution. To prove this it is only necessary to show that all derivatives can be expressed in terms of a finite number of derivatives of ξ_1, ξ_2, ξ_3. (See LIE-ENGEL, *Transformationsgruppen*, vol. I, chap. X.)

For the sake of conciseness write ξ, η, ζ instead of ξ_1, ξ_2, ξ_3 and let $\xi_{i\kappa}$ denote $\partial^2 \xi / \partial x_i \partial x_\kappa$ with other similar abbreviations; also write a, b, c for a_{11}, a_{22}, a_{33}.

We conclude from (5) that

$$\xi_1 - \eta_2, \quad \eta_2 - \zeta_3, \quad a\zeta_1 + c\xi_3, \quad b\zeta_2 + c\eta_3, \quad a\eta_1 + b\xi_2,$$

can be expressed in terms of ξ, η and ζ.

It follows by differentiation that

$$\xi_{111} - \eta_{112}, \quad \eta_{112} - \zeta_{113}, \quad a\zeta_{113} + c\xi_{133}, \quad \xi_{133} - \eta_{233}$$

can each be expressed in terms of derivatives of order not exceeding the second, and, therefore, $a\eta_{112} + c\eta_{233}$ can be so expressed.

Again, $\xi_{122} - \eta_{222}$, $\eta_{222} - \zeta_{223}$, $b\zeta_{223} + c\eta_{233}$, $a\eta_{112} + b\xi_{122}$ can be expressed in terms of derivatives of order not exceeding the second, and, therefore, $a\eta_{112} - c\eta_{223}$ can be so expressed.

We have at once the facts that η_{112} and η_{223} can each be expressed in terms of derivatives of order two at the most, and immediately deduce that

$$\eta_{112}, \quad \xi_{111}, \quad \zeta_{113}, \quad \eta_{233}, \quad \xi_{133}, \quad \zeta_{223}, \quad \eta_{222}, \quad \xi_{122}$$

can each be so expressed.

Once more $a\eta_{13} + b\xi_{23}$, $a\zeta_{12} + c\xi_{23}$, $b\zeta_{12} + c\eta_{13}$ can each be expressed in terms of derivatives of order not exceeding one, and, therefore, so can ξ_{23}, η_{31}, ζ_{12}; it follows that

$$\begin{array}{lll} \xi_{231}, & \eta_{131}, & \zeta_{121}, \\ \xi_{232}, & \eta_{132}, & \zeta_{122}, \\ \xi_{233}, & \eta_{133}, & \zeta_{123} \end{array}$$

can each be expressed in terms of derivatives of order not exceeding two.

We can now see that all derivatives of the third order can be expressed in terms of derivatives of lower order; thus since η_{123} and $\xi_{113} - \eta_{123}$ can be so expressed, ξ_{113} can be; similarly ζ_{133} can be, and, therefore, since $a\zeta_{133} + c\xi_{333}$ can be, ξ_{333} can also; ζ_{123} can be and also $\eta_{122} - \xi_{123}$, therefore η_{122}; from the facts that $\xi_{112} - \eta_{122}$ and $a\eta_{122} + b\xi_{222}$ can be, we then deduce that ξ_{112} and ξ_{222} can be.

We have now proved that all derivatives of the third order of ξ can be expressed in terms of lower derivatives, and similarly we may prove the proposition for η and ζ.

By similar reasoning we may see that there are fifteen linearly independent equations connecting derivatives of the second order, and since there are five equations connecting derivatives of the first order we conclude that all derivatives can be expressed in terms of ξ, η, ζ—four derivatives of the first order and three derivatives of the second order.

Returning now to the equations (5) we see that since a_{11} is not zero we can without altering the equation take it a constant so that $Xa_{11} = 0$; we have then $\rho = 2\,\partial\xi_1/\partial x_1$.

If we now write down the equations which correspond to (3) we see that all derivatives of ζ can be expressed in terms of derivatives of ξ_1, ξ_2, ξ_3, so that we have finally the result that all derivatives of ξ_1, ξ_2, ξ_3, and ζ can be expressed in terms of ξ_1, ξ_2, ξ_3, ζ—four derivatives of the first order and three of the second.

It follows that the values of ξ_1, ξ_2, ξ_3, ζ which satisfy the equations (if such values exist) cannot contain more than eleven arbitrary constants so that the group is of the eleventh order at highest.

It is of course not true that a perfectly general linear partial differential equation admits an infinitesimal transformation at all; what we have proved is that no equation with non-vanishing determinant, whatever its form may be, can admit more than eleven independent transformations. We shall now consider in detail the forms to which such equations can be reduced by point transformations if they admit groups of orders one, two or three. Since every infinitesimal transformation of a group with more than three parameters is contained in at least one group of the third order (LIE, vol. I, p. 593), we shall then have the forms to which all equations of form (1) can be reduced if they admit any group whatever.

§ 3. *Equations which admit one infinitesimal transformation.*

Any linear operator:

$$X \equiv \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \xi_3 \frac{\partial}{\partial x_3} + \zeta z \frac{\partial}{\partial z}$$

becomes

$$X(x_1') \frac{\partial}{\partial x_1'} + X(x_2') \frac{\partial}{\partial x_2'} + X(x_3') \frac{\partial}{\partial x_3'} + X(z') \frac{\partial}{\partial z'},$$

when we apply the transformation:

$$(7) \qquad \begin{aligned} x_i' &= f_i(x_1, x_2, x_3), \\ z' &= zf(x_1, x_2, x_3). \end{aligned}$$

Let now f_2, f_3 be any two independent solutions of

$$\xi_1 \frac{\partial U}{\partial x_1} + \xi_2 \frac{\partial U}{\partial x_2} + \xi_3 \frac{\partial U}{\partial x_3} = 0,$$

and let f_1, f respectively be solutions of

$$\xi_1 \frac{\partial U}{\partial x_1} + \xi_2 \frac{\partial U}{\partial x_2} + \xi_3 \frac{\partial U}{\partial x_3} = 1,$$

$$\xi_1 \frac{\partial U}{\partial x_1} + \xi_2 \frac{\partial U}{\partial x_2} + \xi_3 \frac{\partial U}{\partial x_3} + \zeta U = 0$$

we see that X becomes $\partial/\partial x_1'$.

It follows that any equation which admits X can by a transformation of the form (7) be reduced to an equation which admits $\partial/\partial x_1'$.

It is therefore only necessary to find what conditions the coefficients a_{ik}, a_i and a must satisfy if the corresponding equation admits $\partial/\partial x_1$.

Putting $\xi_1 = 1$, $\xi_2 = \xi_3 = \zeta = 0$ in (2), (3), (4), we get

$$\frac{\partial}{\partial x_1} a_{i\kappa} + \rho a_{i\kappa} = 0 \qquad (i, \kappa = 1, 2, 3),$$

$$\frac{\partial}{\partial x_1} a_i + \rho a_i = 0 \qquad (i = 1, 2, 3),$$

$$\frac{\partial}{\partial x_1} a + \rho a = 0.$$

Now without any loss of generality we can take some one of the coefficients, say $a_{i\kappa}$, to be a constant other than zero (this merely comes to dividing the equation by $a_{i\kappa}$), and therefore $Xa_{i\kappa} = 0$, and consequently $\rho = 0$. It follows that all of the coefficients must have the property of being independent of x_1. Our first result is then that *every linear equation which admits any infinitesimal transformation whatever can be reduced to a form in which all the coefficients depend only on x_2 and x_3, and conversely any such equation will admit at least one transformation.*

§ 4. *Equations admitting groups of the second order.*

Let us now consider the form to which equations can be reduced which admit a group of order two. We know (LIE, vol. III, p. 713) that we can take X_1 and X_2 to be the infinitesimal transformations of such group where X_1 and X_2 are two linear operators such that either $X_1X_2 - X_2X_1 = 0$, or $X_1X_2 - X_2X_1 = X_1$.

Consider first the case where X_1 and X_2 are commutative. Let X_1 be reduced to the form $\partial/\partial x_1$, then the most general linear operator of the form:

$$\xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \xi_3 \frac{\partial}{\partial x_3} + \zeta z \frac{\partial}{\partial z}$$

which is commutative with $\partial/\partial x_1$ is obtained by making ξ_1, ξ_2, ξ_3, ζ depend only on x_2, x_3. Now X_1 is unaltered by any transformation:

$$x_1' = x_1 + f_1(x_2, x_3), \quad x_2' = f_2(x_2, x_3), \quad x_3' = f_3(x_2, x_3), \quad z' = z f(x_2, x_3).$$

Apply such a transformation and X_2 becomes

$$\left(\xi_1 + \xi_2 \frac{\partial f_1}{\partial x_2} + \xi_3 \frac{\partial f_1}{\partial x_3}\right) \frac{\partial}{\partial x_1'} + \left(\xi_2 \frac{\partial f_2}{\partial x_2} + \xi_3 \frac{\partial f_2}{\partial x_3}\right) \frac{\partial}{\partial x_2'}$$
$$+ \left(\xi_2 \frac{\partial f_3}{\partial x_2} + \xi_3 \frac{\partial f_3}{\partial x_3}\right) \frac{\partial}{\partial x_3'} + \left(\xi_2 \frac{\partial f}{\partial x_2} + \xi_3 \frac{\partial f}{\partial x_3} + \zeta f\right) \frac{z'}{f} \frac{\partial}{\partial z'}.$$

If ξ_2 and ξ_3 are both zero, take $f_2 = \xi_1$, and the operator X_2 takes the form:

$$x_2' \frac{\partial}{\partial x_1'} + \zeta z' \frac{\partial}{\partial z'}.$$

If ξ_2 and ξ_3 are not both zero we can choose f_3 and f so that the coefficients of $\partial/\partial x_3'$ and $\partial/\partial z'$ are zero; f_2 so that the coefficient of $\partial/\partial x_2'$ is unity; and f_1 so that the coefficient of $\partial/\partial x_1'$ is zero. We see then that the two operators take one of the forms:

$$X_1 \equiv \frac{\partial}{\partial x_1}, \quad X_2 \equiv \frac{\partial}{\partial x_2};$$

$$X_1 \equiv \frac{\partial}{\partial x_1}, \quad X_2 \equiv x_2 \frac{\partial}{\partial x_1} + \zeta z \frac{\partial}{\partial z}.$$

We can now prove that an equation with non-vanishing determinant cannot admit a group of the latter form.

Since the equation (1) admits X_1 we have seen that the coefficients do not involve x_1; if the equation (1) also admits X_2 we must be able to satisfy the equations (2), (3), (4) by taking $\xi_1 = x_2$, $\xi_2 = 0 = \xi_3 = 0$. Now $Xa_{ik} = 0$ (since a_{ik} does not involve x_1); therefore $\rho a_{22} = \rho a_{33} = 0 = \rho a_{23}$; if $\rho \neq 0$, it follows that $a_{22} = a_{33} = a_{23} = 0$, which would make the determinant zero: if we take $\rho = 0$, from the first of equations (2) we get $a_{12} = 0$, from the fifth $a_{32} = 0$, and from the sixth $a_{22} = 0$; which also makes the determinant zero.

The only permutable group of the second order then which can be admitted is the one which can be reduced to the form $X_1 \equiv \partial/\partial x_1$ $X_2 \equiv \partial/\partial x_2$; and we see as before that all the coefficients must now be independent both of x_1 and x_2.

Similar reasoning will show us that the only types of infinitesimal operators of the group whose composition is $[X_1 X_2] = X_1$ are

$$X_1 \equiv \frac{\partial}{\partial x_1}, \quad X_2 \equiv x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2};$$

$$X_1 \equiv \frac{\partial}{\partial x_1}, \quad X_2 \equiv x_1 \frac{\partial}{\partial x_1} + \zeta z \frac{\partial}{\partial z};$$

where ζ depends only on x_2 and x_3; and by exactly the same kind of reasoning we see that the latter type is inadmissible.

On taking now $\xi_1 = x_1$, $\xi_2 = x_2$, $\xi_3 = 0$, $\zeta = 0$, the equations (2), (3), (4) become:

$$\begin{aligned}
\rho a_{33} + Xa_{33} &= 0, & (\rho - 1)a_{13} + Xa_{13} &= 0, \\
(\rho - 1)a_{23} + Xa_{23} &= 0, & (\rho - 2)a_{11} + Xa_{11} &= 0, \\
(\rho - 2)a_{22} + Xa_{22} &= 0, & (\rho - 2)a_{12} + Xa_{12} &= 0, \\
(\rho - 1)a_1 + Xa_1 &= 0, & (\rho - 1)a_2 + Xa_2 &= 0, \\
\rho a_3 + Xa_3 &= 0, & \rho a + Xa &= 0.
\end{aligned}$$

Here, since all the coefficients are free from x_1, we may write $x_2 \partial/\partial x_2$ for X, and unless $a_{11} = a_{22} = a_{12} = 0$ (a case to be excluded since it makes the determi-

nant zero) we can take one of them constant but not zero, so that $\rho = 2$; it then easily follows that

$$(8)\qquad \begin{gathered} a_{11} = b_{11},\ a_{22} = b_{22},\ a_{33} = b_{33}/x_2^2,\ a_{23} = b_{23}/x_2,\ a_{31} = b_{31}/x_2,\ a_{12} = b_{12}, \\ a_1 = b_1/x_2,\ a_2 = b_2/x_2,\ a_3 = b_3/x_2^2,\ a = b/x_2^2, \end{gathered}$$

where we denote by the coefficients $b_{11}\ b_{22} \cdots b_1 \cdots b$ *functions which are independent both of* x_1 *and* x_2.

§ 5. *Equations which admit the non-integrable group of the third order.*

We pass now to groups of the third order. It is known (LIE, vol. III, p. 716) that the only one which has not a pair of permutable transformations has the composition

$$(\mathrm{I})\qquad (X_1X_2) = X_1, \qquad (X_1X_3) = 2X_2, \qquad (X_2X_3) = X_3.$$

Taking

$$X_1 \equiv \frac{\partial}{\partial x_1},\ X_2 \equiv x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},$$

let

$$X_3 \equiv \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + \xi_3\frac{\partial}{\partial x_3} + z\zeta\frac{\partial}{\partial z}.$$

From the law of composition

$$\frac{\partial}{\partial x_1}\xi_1 = 2x_1, \quad \frac{\partial}{\partial x_1}\xi_2 = 2x_2, \quad \frac{\partial}{\partial x_1}\xi_3 = \frac{\partial}{\partial x_1}\zeta = 0;$$

$$X_2\xi_1 = 2\xi_1, \quad X_2\xi_2 = 2\xi_2, \quad X_2\xi_3 = \xi_3, \quad X_2\zeta = \zeta.$$

And we easily deduce

$$\xi_1 = x_1^2 + x_2^2 f_1(x_3), \quad \xi_2 = 2x_1x_2 + x_2^2 f_2(x_3), \quad \xi_3 = x_2 f_3(x_3), \quad \zeta = x_2 f(x_3).$$

Now by any transformation of the form:

$$x_1' = x_1 + x_2\phi_1(x_3), \quad x_2' = x_2\phi_2(x_3), \quad x_3' = \phi_3(x_3), \quad z' = z\phi(x_3),$$

the forms of X_1 and X_2 are unaltered.

Choose ϕ_1 so that

$$f_1 + f_2\phi_1 + f_3\frac{\partial\phi_1}{\partial x_3} = \phi_1^2.$$

If f_3 is not identically zero we can also choose ϕ_2 and ϕ_3 to satisfy the equations:

$$f_2\phi_2 + f_3\frac{\partial\phi_2}{\partial x_3} = 2\phi_1\phi_2, \qquad f_3\frac{\partial\phi_3}{\partial x_3} = \phi_2;$$

finally choose ϕ so that

$$f_3 \frac{\partial \phi}{\partial x_3} + f\phi = 0,$$

and we see that in the new variables X_1, X_2 are of the same form and X_3 is of the form:

$$x_1^2 \frac{\partial}{\partial x_1} + 2x_1x_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3}.$$

If however $f_3 \equiv 0$ it may be similarly seen by taking $\phi_2 = f_2 - 2\phi_1$, that X_3 can be thrown into the form:

$$x_1^2 \frac{\partial}{\partial x_1} + (2x_1x_2 + x_2^2) \frac{\partial}{\partial x_2} + x_2 f(x_3) z \frac{\partial}{\partial z}.$$

To show that this form cannot be admitted we take $\xi_1 = x_1^2$, $\xi_2 = 2x_1x_2 + x_2^2$, $\xi_3 = 0$. Remembering that $a_{11} = b_{11}$, etc., and that $X = x_1^2 \partial/\partial x_1 + (2x_1x_2 + x_2^2)\partial/\partial x_2$, we have $Xa_{11} = Xa_{22} = Xa_{12} = 0$, and

$$Xa_{13} = -(2x_1 + x_2)a_{13}, \quad Xa_{23} = -(2x_1 + x_2)a_{23}, \quad Xa_{33} = -2(2x_1 + x_2)a_{33}$$

so that the equations (2) become

$$\begin{aligned}
2a_{11}x_2 + 2a_{12}(2x_1 + x_2) &= \rho a_{12}, \\
2a_{13}x_1 &= \rho a_{13} - a_{13}(2x_1 + x_2), \\
2a_{13}x_2 + 2a_{23}(x_1 + x_2) &= \rho a_{23} - a_{23}(2x_1 + x_2), \\
2a_{11}x_1 &= \rho a_{11}, \\
2a_{21}x_2 + 2a_{22}(x_1 + x_2) &= \rho a_{22}, \\
0 &= \rho a_{33} - 2a_{33}(2x_1 + x_2).
\end{aligned}$$

Now if $a_{11} \neq 0$, one has $\rho = 2x_1$, $a_{33} = 0$, and $a_{13} = 0$; from which it follows that $a_{23} = 0$, and the determinant reduces to zero. But if $a_{11} = 0$ and $a_{12} \neq 0$, then $\rho = 2(2x_1 + x_2)$, and by the fifth equation $a_{12}x_2 = a_{22}x_1$; but by (8) a_{12} and a_{22} are independent of x_1 and x_2, and so $a_{12} = 0$, which is contrary to the present hypothesis. Finally take $a_{11} = a_{12} = 0$; if $a_{13} = 0$, the determinant vanishes; buf if $a_{13} \neq 0$, $\rho = 4x_1 + x_2$, and $a_{22} = 0$. In every case then the determinant vanishes, and we conclude that this type cannot be admitted.

Corresponding to the group of composition:

(I) $$(X_1X_2) = X_1, \quad (X_1X_3) = 2X_2, \quad (X_2X_3) = X_3$$

we must then take as our type of infinitesimal transformation:

$$X_3 \equiv x_1^2 \frac{\partial}{\partial x_1} + 2x_1x_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3}.$$

We now have from (2)

$$\begin{aligned}
4\,a_{11}x_1 &= Xa_{11} + \rho a_{11},\\
4\,a_{12}x_2 + 4a_{22}x_1 &= Xa_{22} + \rho a_{22},\\
2\,a_{23} &= Xa_{33} + \rho a_{33},\\
2(a_{13}x_2 + a_{23}x_1) + a_{22} &= Xa_{23} + \rho a_{23},\\
2\,a_{31}x_1 + a_{12} &= Xa_{13} + \rho a_{13},\\
2(a_{11}x_2 + a_{12}x_1) + 2a_{12}x_1 &= Xa_{12} + \rho a_{12}.
\end{aligned}$$

Unless $a_{11} = a_{22} = a_{12} = 0$ (in which case the determinant would vanish) we see by taking $a_{11} =$ constant (or if this vanishes $a_{12} =$ constant, or if both vanish $a_{22} =$ constant) that $\rho = 4x_1$.

Also from (8)

$$\begin{aligned}
Xa_{11} &= x_2\frac{\partial}{\partial x_3}b_{11}, & Xa_{22} &= x_2\frac{\partial}{\partial x_3}b_{22},\\
Xa_{33} &= -4x_1a_{33} + x_2^{-1}\frac{\partial}{\partial x_3}b_{33}, & Xa_{13} &= -2x_1a_{13} + \frac{\partial}{\partial x_3}b_{13},\\
Xa_{23} &= -2x_1a_{23} + \frac{\partial}{\partial x_3}b_{23}, & Xa_{12} &= x_2\frac{\partial}{\partial x_3}b_{12}.
\end{aligned}$$

Hence we deduce:

$$2a_{11} = \frac{\partial b_{12}}{\partial x_3},\quad b_{12} = \frac{\partial b_{13}}{\partial x_3},\quad 4b_{12} = \frac{\partial b_{22}}{\partial x_3},\quad b_{22} + 2b_{13} = \frac{\partial b_{23}}{\partial x_3},\quad 2b_{23} = \frac{\partial b_{33}}{\partial x_3}.$$

Since a_{11} is a constant we find the following solution of these equations:

$$\begin{aligned}
b_{11} &= c_{11},\\
b_{12} &= 2c_{11}x_3 + c_{12},\\
b_{22} &= 4c_{11}x_3^2 + 4c_{12}x_3 + c_{22},\\
b_{13} &= c_{11}x_3^2 + c_{12}x_3 + c_{13},\\
b_{23} &= 2c_{11}x_3^3 + 3c_{12}x_3^2 + (2c_{13} + c_{22})x_3 + c_{23},\\
b_{33} &= c_{11}x_3^4 + 2c_{12}x_3^3 + (2c_{13} + c_{22})x_3^2 + 2c_{23}x_3 + c_{33},
\end{aligned} \tag{9}$$

where we denote by $c_{11}, c_{12} \cdots$ *a system of constants.*

The remaining equations (3) and (4) give after simple reduction:

$$b_{11} = \frac{\partial b_1}{\partial x_3},\quad 2b_{12} + 2b_1 = \frac{\partial b_2}{\partial x_3},\quad b_2 = \frac{\partial b_3}{\partial x_3},\quad 0 = \frac{\partial b}{\partial x_3};$$

and, therefore,

$$\begin{aligned}
b_1 &= c_{11}x_3 + c_1,\\
b_2 &= 3c_{11}x_3^2 + 2(c_{12} + c_1)x_3 + c_2,\\
b_3 &= c_{11}x_3^3 + (c_{12} + c_1)x_3^2 + c_2x_3 + c_3,\\
b &= c,
\end{aligned} \tag{10}$$

where c, c_1, c_2, c_3 denote *constants.*

§ 6. *Equations admitting other groups of the third order.*

The only other groups of the third order are (LIE, vol. III, p. 716) those which have the compositions:

(II) $(X_1X_2)=0,\quad (X_1X_3)=X_1,\quad (X_2X_3)=mX_2$ (m constant);

(III) $(X_1X_2)=0,\quad (X_1X_3)=X_1,\quad (X_2X_3)=X_1+X_2$;

(IV) $(X_1X_2)=0,\quad (X_1X_3)=0\ ,\quad (X_2X_3)=X_1$;

(V) $(X_1X_2)=0,\quad (X_2X_3)=0\ ,\quad (X_1X_3)=0$.

Taking (II) first we have seen that we may take $X_1\equiv\partial/\partial x_1$, $X_2\equiv\partial/\partial x_2$; if $X_3\equiv\xi_1\,\partial/\partial x_1+\xi_2\,\partial/\partial x_2+\xi_1\,\partial/\partial x_3+\zeta z\,\partial/\partial z$, we have from the composition of the group:

$$\xi_1=x_1+f_1(x_3),\quad \xi_2=mx_2+f_2(x_3),\quad \xi_3=f_3(x_3),\quad \zeta=f(x_3).$$

Noticing that X_1 and X_2 are unaltered in form by any transformation:

$$x_1'=x_1+\phi_1(x_3),\quad x_2'=x_2+\phi_2(x_3),\quad x_3'=\phi_3(x_3),\quad z'=z\phi(x_3),$$

we see that by properly choosing the forms ϕ_1, ϕ_2, ϕ_3, ϕ, we can without altering the forms of X_1 and X_2 throw X_3 into one of the forms:

(α) $$x_1\frac{\partial}{\partial x_1}+mx_2\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3};$$

(β) $$x_1\frac{\partial}{\partial x_1}+mx_2\frac{\partial}{\partial x_2}+f(x_3)z\frac{\partial}{\partial z}.$$

In (α), remembering that all the coefficients a_{hk}, a_h, a now depend only on x_3, we see that (2), (3), 4 become:

(11)
$$\left\{\begin{array}{ll}
2a_{11}=\rho a_{11}+\dfrac{\partial a_{11}}{\partial x_3}, & 2ma_{22}=\rho a_{22}+\dfrac{\partial a_{22}}{\partial x_3},\\[2ex]
0=\rho a_{33}+\dfrac{\partial a_{33}}{\partial x_3}, & ma_{23}=\rho a_{23}+\dfrac{\partial a_{23}}{\partial x_3},\\[2ex]
a_{31}=\rho a_{13}+\dfrac{\partial a_{13}}{\partial x_3}, & (m+1)a_{12}=\rho a_{12}+\dfrac{\partial a_{12}}{\partial x_3},\\[2ex]
a_1=\rho a_1+\dfrac{\partial a_1}{\partial x_3}, & ma_2=\rho a_2+\dfrac{\partial a_2}{\partial x_3},\\[2ex]
0=\rho a_3+\dfrac{\partial a_3}{\partial x_3}, & 0=\rho a+\dfrac{\partial a}{\partial x_3}.
\end{array}\right.$$

Now first suppose that a_{33} is not zero; then we take it a constant c_{33} and we get the solution of (9) given in the first column of the following table:

$a_{33} = c_{33}$	$a_{33} = 0$	$a_{33} = 0$
$a_{11} = c_{11}e^{2x_3}$	$a_{11} = c_{11}e^{(2-m)x_3}$	$a_{11} = c_{11}e^{(2-2m)x_3}$
$a_{22} = c_{22}e^{2mx_3}$	$a_{22} = c_{22}e^{mx_3}$	$a_{22} = a_{22}$
$a_{23} = c_{23}e^{mx_3}$	$a_{23} = c_{23}$	$a_{23} = 0$
$a_{31} = c_{31}e^{x_3}$	$a_{31} = c_{31}e^{(1-m)x_3}$	$a_{31} = c_{13}e^{(1-2m)x_3}$
$a_{12} = c_{12}e^{(1+m)x_3}$	$a_{12} = c_{12}e^{x_3}$	$a_{12} = c_{12}e^{(1-m)x_3}$
$a_1 = c_1e^{x_3}$	$a_1 = c_1e^{(1-m)x_3}$	$a_1 = c_1e^{(1-2m)x_3}$
$a_2 = c_2e^{mx_3}$	$a_2 = c_2$	$a_2 = c_2e^{-mx_3}$
$a_3 = c_3$	$a_3 = c_3e^{-mx_3}$	$a_3 = c_3e^{-2mx_3}$
$a = c$	$a = ce^{-mx_3}$	$a = ce^{-2mx_3}$
$\rho = 0$	$\rho = m$	$\rho = 2m$

where we always use the letters $c_{11}\ c_{12} \cdots c_1 \cdots c$ to denote constants.

If $a_{33} = 0$ but $a_{23} \neq 0$ then we obtain the solution given in the second column. If $a_{33} = a_{23} = 0$ then we cannot have $a_{22} = 0$ (since then the determinant would vanish) and the only other solution is given in the third column. Notice that of these three solutions the second and the third are really included under the first and do not give distinct forms of (1).

Take now the case when X_3 is of type (β); we have

$$\rho a_{33} = 0, \qquad \rho a_{23} = m a_{23},$$

$$\rho a_{13} = a_{13}, \qquad \rho a_{22} = 2m a_{22},$$

$$\rho a_{11} = 2a_{11}, \qquad \rho a_{12} = (1+m) a_{12},$$

$$\rho a_1 = a_1 - a_{13}\frac{\partial f}{\partial x_3}, \qquad \rho a_2 = m a_2 - a_{23}\frac{\partial f}{\partial x_3},$$

$$\rho a_3 = -a_{33}\frac{\partial f}{\partial x_3}, \qquad \rho a = -2a_3\frac{\partial f}{\partial x_3} - a_{33}\frac{\partial^2 f}{\partial x_3^2}.$$

And it is not difficult to see that the only solutions which do not cause the determinant to vanish are the set given in the table of §7 under II(β). Notice that the general form of II(β) is not admitted, but only the cases when $f(x_3)$ is a constant, and $m = -1, 2,$ or $\frac{1}{2}$.

§ 7. *Remaining groups of the third order and general conclusions.*

We may similarly prove that the groups (III) can be thrown into one of the two forms:

$$(\alpha) \qquad X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = (x_1 + x_2)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3};$$

$$(\beta) \qquad X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = (x_1 + x_2)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + f(x_3)z\frac{\partial}{\partial z}.$$

And it may be shown by methods similar to those already employed that the latter form cannot be admitted by a differential equation with non-vanishing determinant.

The first gives:

$$2(a_{11} + a_{12}) = \rho a_{11} + \frac{\partial a_{11}}{\partial x_3}, \qquad 2a_{22} = \rho a_{22} + \frac{\partial a_{22}}{\partial x_3},$$

$$0 = \rho a_{33} + \frac{\partial a_{33}}{\partial x_3}, \qquad 2a_{12} + a_{22} = \rho a_{12} + \frac{\partial a_{12}}{\partial a_3},$$

$$a_{13} + a_{23} = \rho a_{13} + \frac{\partial a_{13}}{\partial x_3}, \qquad a_{23} = \rho a_{23} + \frac{\partial a_{23}}{\partial x_3},$$

$$a_1 + a_2 = \rho a_1 + \frac{\partial a_1}{\partial x_3}, \qquad a_2 = \rho a_2 + \frac{\partial a_2}{\partial x_3},$$

$$0 = \rho a_3 + \frac{\partial a_3}{\partial x_3}, \qquad 0 = \rho a + \frac{\partial a}{\partial x_3}.$$

And of these equations we have the solution III(α) of the table.

The group (IV) has two possible types:

$$(\alpha) \qquad X_1 \equiv \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = x_2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3};$$

$$(\beta) \qquad X_1 \equiv \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + f(x_3)z\frac{\partial}{\partial z}.$$

Type (α) leads to

$$\rho a_{33} + \frac{\partial a_{33}}{\partial x_3} = 0, \qquad \rho a_{22} + \frac{\partial a_{22}}{\partial x_3} = 0,$$

$$\rho a_{23} + \frac{\partial a_{23}}{\partial x_3} = 0, \qquad \rho a_{13} + \frac{\partial a_{13}}{\partial x_3} = a_{23},$$

$$\rho a_{12} + \frac{\partial a_{12}}{\partial x_3} = a_{22}, \qquad \rho a_{11} + \frac{\partial a_{11}}{\partial x_3} = 2a_{12},$$

$$\frac{\partial a_1}{\partial x_3} = a_2, \qquad \frac{\partial a_2}{\partial x_3} = 0,$$

$$\frac{\partial a_3}{\partial x_3} = 0, \qquad \frac{\partial a}{\partial x_3} = 0.$$

Of these the most general solution of non-vanishing determinant is found under IV(a).

Type (β) leads to

$$\rho a_{33} = 0\,, \qquad \rho a_{23} = a_{33}\,,$$
$$\rho a_{22} = 2a_{23}\,, \qquad \rho a_{13} = a_{23}\,,$$
$$\rho a_{12} = a_{22} + a_{13}\,, \qquad \rho a_{11} = 2a_{12}\,,$$
$$a_2 = a_{13}\frac{\partial f}{\partial x_3} + \rho a_1\,, \qquad a_3 = a_{23}\frac{\partial f}{\partial x_3} + \rho a_2\,,$$
$$0 = a_{33}\frac{\partial f}{\partial x_3} + \rho a_{33}\,, \qquad 0 = a_{33}\frac{\partial^2 f}{\partial x_3^2} + 2a_3\frac{\partial f}{\partial x_3} + \rho a\,,$$

of which the solution is given under IV(β).

Finally the only type of group (V) which could be admitted is $X_1 = \partial/\partial x_1$, $X_2 = \partial/\partial x_2$, $X_3 = \partial/\partial x_3$. We see this by noticing that when we bring X_1 and X_2 into the forms $\partial/\partial x_1$ and $\partial/\partial x_2$ respectively, then X_3 is from the laws of composition of the form:

$$f_1(x_3)\frac{\partial}{\partial x_1} + f_2(x_3)\frac{\partial}{\partial x_2} + f_3(x_3)\frac{\partial}{\partial x_3} + f(x_3)z\frac{\partial}{\partial z}\,.$$

If $f_3(x_3) \equiv 0$, it may easily be verified, by taking $x_3' = f_1(x_3)$ and throwing X_3 into the form $x_3\,\partial/\partial x_1 + f_2(x_3)\,\partial/\partial x_2 + f(x_3)z\,\partial/\partial z$, that the group cannot be admitted without causing the determinant to vanish.

If $f_3(x_3) \neq 0$, by a transformation of the form:

$$x_1' = x_1 + \phi_1(x_3)\,, \quad x_2' = x_2 + \phi_2(x_3)\,, \quad x_3' = \phi_3(x_3)\,, \quad z' = z\phi(x_3)\,,$$

we can, without altering the forms of X_1 and X_2, throw X_3 into the form $\partial/\partial x_3$.

The necessary and sufficient condition that (1) should admit such a group is then that by point-transformation of the form (7) it can be thrown into the form of a linear partial differential equation with constant coefficients.

We can now tabulate the results at which we have arrived in this paper. If the equation:

$$a_{11}z_{11} + a_{22}z_{22} + a_{33}z_{33} + 2a_{23}z_{23} + 2a_{31}z_{31} + 2a_{12}z_{12} + 2a_1z_1 + 2a_2z_2 + 2a_3z_3 + az = 0$$

of non-vanishing determinant, admits at least one infinitesimal transformation, it can be reduced by point transformation to such a form that all the coefficients are independent of the variable x_1. If it admits at least two such transformations it can be reduced to one of two forms, the first form being that in which the coefficients are independent of x_1 and x_2, and the second that in which

$$a_{11} = b_{11}\,, \quad a_{22} = b_{22}\,, \quad a_{33} = b_{33}/x_2^2\,, \quad a_{23} = b_{23}/x_2\,, \quad a_{31} = b_{31}/x_2\,, \quad a_{12} = b_{12}\,,$$
$$a_1 = b_1/x_2\,, \quad a_2 = b_2/x_2\,, \quad a_3 = b_3/x_2^2\,, \quad a = b/x_2^2$$

where all the quantities b_{ik}, b_i, b are independent of x_1 and x_2.

An equation which admits at least three infinitesimal transformations can be reduced to some one of the forms given in the accompanying table, where the number at the head of a column indicates the group corresponding to the equation given in that column, and where the b_{ik}, b_i, b are independent of x_1 and x_2, and the c_{ik}, c_i, c are constants. I hope in some future paper to discuss the further relations necessary between the constants $c_{ik}\cdots$ and the functions $b_{ik}\cdots$ in order that the equation may admit groups of order above the third.

Coefficient of	I	II(α)	II(β)	II(β)	II(β)
z_{11}	c_{11}	$c_{11}e^{2x_3}$	0	b_{11}	0
z_{22}	$4c_{11}x_3^2 + 4c_{12}x_3 + c_{22}$	$c_{22}e^{2mx_3}$	0	0	b_{22}
	$\{c_{11}x_3^4 + 2c_{12}x_3^3 + (c_{22} + 2c_{13})x_3^2 + 2c_{23}x_3 + c_{33}\} \div x_2^2$	c_{33}	b_{33}	0	0
$2z_{23}$	$\{2c_{11}x_3^3 + 3c_{12}x_3^2 + (c_{22} + 2c_{13})x_3 + c_{23}\} \div x_2$	$c_{23}e^{mx_3}$	0	b_{23}	0
$2z_{31}$	$(c_{11}x_3^2 + c_{12}x_3 + c_{13}) \div x_2$	$c_{31}e^{x_3}$	0	0	b_{13}
$2z_{12}$	$2c_{11}x_3 + c_{12}$	$c_{12}e^{(1+m)x_3}$,	b_{12}	0	0
$2z_1$	$(c_{11}x_3 + c_1) \div x_2$	$c_1e^{x_3}$	0	0	b_1
$2z_2$	$\{3c_{11}x_3^2 + 2(c_{12} + c_1)x_3 + c_2\} \div x_2$	$c_2e^{mx_2}$	0	b_2	0
$2z_3$	$\{2c_{11}x_3^3 + 2(c_{12} + c_1)x_3^2 + 2c_2x_3 + c_3\} \div x_2^2$	c_3	b_3	0	0
z	$c \div x_2^2$	c	b	0	0
			$m=-1$	$m=2$	$m=\frac{1}{2}$

Coefficient of	III(α)	IV(α)	IV(β)	V
z_{11}	$(c_{22}x_3^2 + 2c_{12}x_3 + c_{11})e^{2x_3}$	$c_{22}x_3^2 + 2c_{12}x_3 + c_{11}$	b_{11}	c_{11}
z_{22}	$c_{22}e^{2x_3}$	c_{22}	b_{22}	c_{22}
z_{33}	c_{33}	c_{33}	0	c_{33}
$2z_{23}$	$c_{23}e^{x_3}$	c_{23}	0	c_{23}
$2z_{31}$	$(c_{22}x_3 + c_{13})e^{x_3}$	$c_{22}x_3 + c_{13}$	$-b_{22}$	c_{31}
$2z_{12}$	$(c_{22}x_3 + c_{12})e^{2x_3}$	$c_{22}x_3 + c_{12}$	0	c_{12}
$2z_1$	$(c_2x_3 + c_1)e^{x_3}$	$c_2x_3 + c_1$	b_1	c_1
$2z_2$	$c_2e^{x_3}$	c_2	$-b_{22}\dfrac{\partial f(x_3)}{\partial x_3}$	c_2
$2z_3$	c_3	c_3	0	c_3
z	c	c	b	c

HERTFORD COLLEGE, OXFORD,
December 7, 1899.